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A Contribution to Automated-oriented Reasoning about Permutability of Sequent Calculi Rules

Tatjana Lutovac[1] and James Harland[2]

[1] Department of Applied Mathematics, Faculty of Electrical Engineering,
University of Belgrade, P.O. Box 35-54, 11120 Belgrade, Serbia
tlutovac@eunet.rs
[2] School of Computer Science and Information Technology, RMIT University
GPO Box 2476V, Melbourne, 3001, Australia
james.harland@rmit.edu.au

Abstract. Many important results in proof theory for sequent calculus (cut-elimination, completeness and other properties of search strategies, etc) are proved using permutations of sequent rules. The focus of this paper is on the development of systematic and automated-oriented techniques for the analysis of permutability in some sequent calculi. A representation of sequent calculi rules is discussed, which involves greater precision than previous approaches, and allows for correspondingly more precise and more general treatment of permutations. We define necessary and sufficient conditions for the permutation of sequence rules. These conditions are specified as constraints between the multisets that constitute different parts of the sequent rules. The authors extend their previous work in this direction to include some special cases of permutations.

Keywords: Automated reasoning, permutation, sequent calculi, proof search, linear logic.

1. Introduction

Sequent calculus is a well-known analytical tool for the design, specification, and implementation of a number of computational systems [20]. Furthermore, many important results for the sequent calculus (cut-elimination, completeness of search strategies, etc) have been proved using permutations of sequent rules. The permutability of the sequent rules has been thoroughly studied and the results effectively applied to formulate various proof search strategies (focusing [1], uniformity [21], normal-form [10]). It is notable that many proof search strategies can be expressed as restrictions on the order of application of the rules of the sequent calculus. The properties of these strategies are then shown by permutation arguments, such as demonstrating that a given strategy is complete by transforming an arbitrary proof into one which obeys the strategy. Also, formalised proofs of cut admissibility sometimes rely on the invertibility of the rules of a sequent calculus [3, 5].

Recently, much attention has been devoted to linear logic as a general metatheory for specification of different properties for variety of sequent calculus systems [22], [23]. It is shown that linear logic can be successfully used as a meta-logic that capture a number of certain important properties of encoded proof systems. However, there has been comparatively little effort on the development of systematic and automation-oriented techniques (for permutation analysis of sequent rules) which are not based on a particular selected meta-logic.

Ciabattoni and Terui [5] have given a syntactical characterization of cut-elimination for single-conclusioned sequent calculus. Chapman [3] has given syntactic sufficient criteria for invertibility of sequent rules are given, as a means to automating cut admissibility proofs. In both cases a schematic representation of sequent rules are considered, rather than direct applications of rules in a derivation. In our case, we give conditions for permutability that are weaker than invertibility, and for more general systems than single-conclusioned ones. In particular, our approach analyzes the context of a particular derivation in which permutation may be possible, rather than stronger 'context-free' conditions such as invertibility.

Permutations of inference rules can be treated in a general way by distinguishing the formulae roles in the inference rules. The research reported in this paper was motivated by the fact that the existing approaches in permutation analysis ([10, 13]) cannot detect and 'explain' all possible permutations and are too implicit from an automation perspective. Generally, checking whether two consecutive inference rules can be permuted in a given proof is a tedious task, which is usually performed by hand, and is ripe for automation.

It should be noted that our approach is to enable as many permutations as possible, which includes what may be termed 'conditional permutations'. For example, cut-elimination results are often proved by showing that the cut rule in a given system can be permuted upwards (i.e. towards the leaves) under all possible circumstances, *i.e.* that for any inference rule I, any proof in which I is immediately followed by the cut rule can be permuted to one in which the cut rule appears closer to the leaves. In this paper, we generalise this idea to arbitrary rules I and J, which may have conditions under which the permutation is possible, and some in which it is not.

In [16] the present authors proposed a representation of sequent calculi rules which involves greater precision than previous approaches. This involves a finer-grained specification of sequent rules, which allows the development of a syntactical characterization of the conditions under which proofs can be rearranged. As the class of possible sequent rules is very broad, we restrict our attention to sequent calculi in which contexts are commutative and associative, and contain only a single zone, and that rules have no side conditions and at most two premises.[3]

This paper builds on the previous work of the authors reported in [16], adding a case study in permutation analysis. This includes a number of special cases

[3] The assumptions about context mean that we can use multisets of formulae as our basic structure for sequents. The assumptions about the number of premises and no side effects are very common in sequent systems.

of permutations, as well as more subtler proof rearrangements when permutations are not strictly possible. Two types of results are developed: (1) necessary and sufficient conditions for the permutation of inference rules in a given proof and (2) properties of the possible forms of proofs after permutation. This paper attempts to enhance the general setting for automated-oriented study of the permutability of inference rules in the sequent calculus. The work is intended as a contribution to the library of automated support tools for reasoning about permutations in sequent calculi proof search. It is believed that the proposed specifications can be implemented and utilised by means of an automated proof assistant such as Twelf [25].

This paper is organised as follows. Section 2 recalls the basic notions and the motivation to refine the existing framework for permutation analysis; it also contains a brief overview of the proposed refined specification of sequent rules and several applications. Section 3 further extends and generalises the applications of the refined sequent structure. Finally, Section 4 concludes the paper and provides an indication of future work.

2. A Standard Approach in Permutation Analysis

2.1. Some Basic Notions

Rules in sequent calculi are defined in terms of *metavariables*, which may stand for either a single formula, or an arbitrary number of formulae. This makes it possible to write a single rule which will generally have an infinite number of instances, each corresponding to a particular instantiation of the metavariables to specific formulae. In this paper we will use either English letters (e.g. A, $!C$, F, a, p, q) or lower-case Greek letters (e.g.ϕ,$\psi \otimes \phi$) to represent a single formulae, and upper-case Greek letters (e.g. Γ, Δ) to represent an arbitrary number of formulae. For example, the rule $\otimes R$ is below, followed by three instances of it.

$$\frac{\Gamma_1 \vdash \phi, \Delta_1 \quad \Gamma_2 \vdash \psi, \Delta_2}{\Gamma_1, \Gamma_2 \vdash \phi \otimes \psi, \Delta_1, \Delta_2} \otimes R$$

$$\frac{p \vdash p \quad q \vdash q}{p, q \vdash p \otimes q} \otimes R \qquad \frac{r, p \vdash r, q \quad s \vdash s}{r, s, p \vdash r \otimes s, q} \otimes R \qquad \frac{r, p \vdash r, q \quad s, t \vdash s, v}{r, s, p, t \vdash r \otimes s, q, v} \otimes R$$

In the first instance, $\Gamma_1 = \{p\}$, $\Gamma_2 = \{q\}$, $\Delta_1 = \emptyset$, $\Delta_2 = \emptyset$, ψ is p and ϕ is q. In the second instance, $\Gamma_1 = \{r, p\}$, $\Gamma_2 = \{s\}$, $\Delta_1 = \{q\}$, $\Delta_2 = \emptyset$, ψ is r and ϕ is s. In the third instance, $\Gamma_1 = \{r, p\}$, $\Gamma_2 = \{s\}$, $\Delta_1 = \{q\}$, $\Delta_2 = \{v\}$, ψ is r and ϕ is s.

This means that when analyse rules, the objects that are the subject of our analysis are metavariables, which, as above, we distinguish into two types: those which represent a *single formula*, and those which represent *an arbitrary number of formulae*. There are some rules which only apply to particular atomic formulae, and so we also need to allow for these in our rules.

The general form of a rule in the sequent calculus is then $\frac{Premises}{Conclusion}$ where $Conclusion$ consists of exactly one sequent, and $Premises$ can be either

0, 1, or 2 sequents. Each sequent is of the form $X_1, \ldots, X_n \vdash Y_1, \ldots, Y_m$ where X_i and Y_j can be either a logical constant (such as **1**, $\top$ or $\bot$), a single-formula metavariable, or an arbitrary-formula metavariable. Note that these are the only two types of meta-variable permitted in a sequent, and, as above, it is always possible to syntactically distinguish the two types. Note also that it is convenient to construct *metaformulae* from metavariables. For example, consider $\phi \otimes \psi$ in the $\otimes R$ rule above. This is a metavariable of a particular form, but its main importance is to allow us to specify the relationships between the formulae in the premises (ϕ in one premise and ψ in the other) and in the conclusion ($\phi \otimes \psi$). Hence a metaformula is either an atomic formula, a metavariable, or constructed from these and the logical connectives. The notion of a metaformula can be easily extended to submetaformulae in the obvious way, e.g. in the $\otimes R$ rule above, ϕ and ψ are submetaformulae of the metaformula $\phi \otimes \psi$.

A distinction is made between rules and their instances, which we refer to as *inferences*. Most of the presented results and specifications are about rules (*i.e.* about the restrictions and relations which need to be fulfilled after the instantiation of a rule), and these are proved by showing the result for arbitrary instances of the rule. Inferences may contain no metaformulae, as they have all been instantiated with formulae. However, metaformulae and their instantiations (as well as metamultisets and their instantiations) will be denoted in the same way; it will always be possible to disambiguate such annotations from the context. Different occurrences of the same metaformula in a rule are also distinguished.

A *derivation* of a sequent $\Gamma \vdash \Delta$ is a tree whose nodes are labelled with sequents such that the root node is labelled with $\Gamma \vdash \Delta$ (which is then called the *end-sequent* of the derivation) and the internal nodes and the leaf nodes are instances of one of the sequent rules. A *proof* of a sequent $\Gamma \vdash \Delta$ is a derivation of sequent $\Gamma \vdash \Delta$ whose leaf nodes are labelled with initial rules (*i.e.* rules with no premises). A sequent $\Gamma \vdash \Delta$ is *provable* in the sequent calculus formalization of a logic (or logical fragment) if there is a proof with $\Gamma \vdash \Delta$ as the end-sequent.

The *permutation* of two adjacent inference rules of a given proof is reversing their order in the proof but without disturbing the rest of the proof (the parts above and below the inferences modulo duplication of some proof branches and renaming certain free variables) as a result of which a proof *equivalent* to the given proof is derived.

Most of the examples presented in this paper come from one-sided linear logic (LL). Linear logic [11] is a refinement of classical logic, in that there is a fragment of linear logic which has precisely the same properties as classical logic; at the same time however, linear logic contains features which are not present in classical logic. In essence, these features are due to removing the rules for contraction and weakening and re-introducing them in a controlled manner. A linear formula, written ϕ, represents a resource and must be used *exactly once* in a linear proof derivation. An exponential formula $!\phi$ or $?\phi$, is like a formula of classical logic and may be used as often as needed, and may be ignored if desired, in a proof construction.

There is vast literature on linear logic and its variants, such that no attempt is made here to provide a detailed general introduction. However, Figure 1 below shows the sequent calculus rules for one-sided sequent calculus[4] in linear logic (LL).

Fig. 1. One-sided Sequent Calculus for Linear Logic

$$
\frac{}{\vdash \phi, \phi^{\perp}}\ axiom
\qquad
\frac{\vdash \phi, \Delta \quad \vdash \phi^{\perp}, \Delta^{'}}{\vdash \Delta, \Delta^{'}}\ cut
\qquad
\frac{}{\vdash \mathbf{1}}\ 1
\qquad
\frac{\vdash \Delta}{\vdash \perp, \Delta}\ \perp
\qquad
\frac{}{\vdash \top, \Delta}\ \top
$$

$$
\frac{\vdash \phi, \Delta \quad \vdash \psi, \Delta^{'}}{\vdash \phi \otimes \psi, \Delta, \Delta^{'}}\ \otimes
\quad
\frac{\vdash \phi, \Delta \quad \vdash \psi, \Delta}{\vdash \phi \& \psi, \Delta}\ \&
\qquad
\frac{\vdash \phi_i, \Delta}{\vdash \phi_1 \oplus \phi_2, \Delta}\ \oplus_i,\ i = 1, 2
$$

$$
\frac{\vdash \phi, \psi, \Delta}{\vdash \phi \wp \psi, \Delta}\ \wp
\qquad
\frac{\vdash \phi^{\perp}, \psi, \Delta}{\vdash \phi \multimap \psi, \Delta}\ \multimap
\qquad
\frac{\vdash \phi, ?\Delta}{\vdash !\phi, ?\Delta}\ !
\qquad
\frac{\vdash \phi, \Delta}{\vdash ?\phi, \Delta}\ ?
$$

$$
\frac{\vdash \Delta}{\vdash ?\phi, \Delta}\ w
\qquad
\frac{\vdash ?\phi, ?\phi, \Delta}{\vdash ?\phi, \Delta}\ c?
\qquad
\frac{\vdash \phi[y/x], \Delta}{\vdash \forall x.\phi, \Delta}\ \forall^{*}
\quad
\frac{\vdash \phi[t/x], \Delta}{\vdash \exists x.\phi, \Delta}\ \exists
$$

* y is not free in Δ

2.2. The Existing Framework: Problems and Some Solutions

The problem of permutation analysis is not new ([13, 10]). Let us begin with a brief description of the terminology and approach given in [10]. The *active* formulae of an inference are the formulae which are present in the premise(s), but not in the conclusion. The *principal* formula of an inference is the formula which is present in the conclusion, but not in the premise(s). The *context* formulae are those which are neither active nor principal. Intuitively, the inference converts the active formulae into the principal formula, leaving context formulae unchanged.

Let $\begin{matrix}[\mathcal{D}]\\ I\end{matrix}$ denote a derivation in which the end-sequent of derivation $\mathcal{D}$ is a premise of a rule I and let $\begin{matrix}[\mathcal{D}]\\ I\\ J\end{matrix}$ denote a derivation in which the end-sequent

[4] One-sided sequents of the form $\vdash \Delta$ are used, where Δ is a multiset of formulae. For the multisets Δ, and Δ_1 let $\Delta \backslash \Delta_1$ denote the difference of multisets Δ and Δ_1 (so $\{a, a, b, b, b, c\} \backslash \{a, b, b, c, d\} = \{a, b\}$) and let Δ, Δ_1 denote the sum of multisets Δ and Δ_1 (so, $\{a, a, b\}, \{a, b, b, c\} = \{a, a, a, b, b, b, c\}$).

of derivation $\overset{[\mathcal{D}]}{I}$ is a premise of a rule J. Note that both I and J here could be either unary or binary rules. In the cases where J is a binary rule, I can occur in either the left or right premise of J.

Let $\begin{array}{c}\vdots\,\mathcal{D}\\ \alpha\end{array}$ denote the derivation $\mathcal{D}$ with the end-sequent α, and let $\begin{array}{c}\vdots\,\mathcal{D}\\ \Gamma\vdash\Delta\end{array}$ denote the derivation $\mathcal{D}$ with the end-sequent $\Gamma\vdash\Delta$.

When looking to swap the order of two adjacent inferences (*i.e. to permute an inference I over an inference J*) in a given derivation $\begin{array}{c}[\mathcal{D}]\\ I\\ J\end{array}$ it is sufficient and necessary to check the following conditions ([10]) :

- **A.** I and J are in the *permutation position*: the principal formula of the upper inference I is not an active formula of the lower inference J[5].
- **B_1.** J is applicable on the appropriate premise(s) of I;
- **B_2.** The end-sequent of $\overset{[\mathcal{D}]}{J}$ can be a premise of I;
- **C.** Derivations $\begin{array}{c}[\mathcal{D}]\\ I\\ J\end{array}$ and $\begin{array}{c}[\mathcal{D}]\\ J\\ I\end{array}$ have the same end-sequent and the same leaves modulo duplication of some of them and renaming certain free variables.

For example, consider the inferences below. In either inference, the categorization is as specified in the table shown on the right-hand side. Note that in the left-hand subproof $\neg L$ and $\otimes L$ are not in the GP permutation position, while $\neg L$ and $\wp R$ are in the GP permutation position.

$$\dfrac{\dfrac{\dfrac{q\vdash p,q,r}{q\vdash p,q\wp r}\,\wp R}{\neg p,q\vdash q\wp r}\,\neg L}{\neg p\otimes q\vdash q\wp r}\,\otimes L \quad\longmapsto\quad \dfrac{\dfrac{\dfrac{q\vdash p,q,r}{\neg p,q\vdash q,r}\,\neg L}{\neg p,q\vdash q\wp r}\,\wp R}{\neg p\otimes q\vdash q\wp r}\,\otimes L$$

Rule	Principal Formula	Active Formulae
$\otimes L$	$\neg p\otimes q$	$\neg p, q$
$\neg L$	$\neg p$	p
$\wp R$	$q\wp r$	q, r

As illustrated by the following example, two adjacent rules in the GP permutation position are not necessarily permutable. In the proof below the rules $?$ and $!$ are in the GP permutation position but they are not permutable. The application of sequent rules may involve some restrictions on the structure of some formulae from a rule's premise and/or conclusion (for example, a particular top-most connective may be required). As the linear logic rule $!$ requires all non-active formulae from the premise to be prefixed by $?$, the condition B_2 is not fulfilled:

$$\dfrac{\dfrac{\vdash A,\ B,\ ?\Delta}{\vdash A,\ ?B,\ ?\Delta}\,?}{\vdash !A,\ ?B,\ ?\Delta}\,! \quad\not\longmapsto\quad \dfrac{\vdash A,\ B,\ ?\Delta}{\times}\,!$$

The existing analysis of permutations via the notions of active and principal formulae is sometimes too coarse to capture when permutations are possible. There are many situations where the permutations cannot be explained by the

[5] Referred to here as the *GP (due to Galmiche and Perrier) permutation position* .

existing framework *i.e.* there are inference pairs (say I followed by J) which can be permuted but the conditions $\boldsymbol{A, B_1, B_2}$ and $\boldsymbol{C}$ are not fulfilled. Two examples are provided below (some interesting examples may be found in [15]).

Example 1 Consider the following rule from the LJ sequent calculus as well as from a multiple-conclusioned LJ ([6]) calculus: $\dfrac{\Gamma, a, F \vdash G}{\Gamma, a, a \to F \vdash G} \to L$.
This results in the following categorization:

Principal Formula	Active Formulae	Context
$a \to F$	F	Γ, G

What can be said about the (atomic) formula a? According to the existing framework, a is a context formula. However it has a specific 'role' that should not characterize any context formula, which is that the occurrence of a in the premise of the rule $\to L$ is necessary for the rule to be applied and hence a cannot be omitted or freely replaced with another formula. Another possibility is to interpret the occurrences of the formula a in the premise and in the conclusion as an active and a principal formula, respectively. Under such an interpretation, for example, in the derivation below, among formulae $a \to F_1,\ a \to F_2$ the choice of the formula to be decomposed next will not be arbitrary. This means that some possible permutations cannot be detected: the rule instances are not in a permutation position as a is both a principal formula of the upper $\to L$ inference and an active formula of the lower $\to L$ inference.

$$\dfrac{\dfrac{\Gamma, a, F_1, F_2, \Delta \vdash G}{\Gamma, a, a \to F_1, F_2, \Delta \vdash G} \to L}{\Gamma, a, a \to F_1, a \to F_2, \Delta \vdash G} \to L \qquad \leftrightarrow \qquad \dfrac{\dfrac{\Gamma, a, F_1, F_2, \Delta \vdash G}{\Gamma, a, F_1, a \to F_2, \Delta \vdash G} \to L}{\Gamma, a, a \to F_1, a \to F_2, \Delta \vdash G} \to L$$

The intuition behind this is that a formula occurrence whose presence in a premise is necessary (but not sufficient) for the rule application, and which is copied unchanged into the conclusion should be considered and treated separately from the context, active and principal parts of the rule. Such formulae are referred to as *quasi-active* formulae in this paper.

Example 2 Consider now the macro-rule $\sharp$, which is a result of fusion of the $\wp$ and w rules from linear logic, and the permutation below:

$$\underset{\text{schema-rule}}{\dfrac{\vdash \phi, \psi, \Gamma}{\vdash \phi\sharp\psi, \Gamma, ?\delta}\ \sharp} \qquad \dfrac{\dfrac{\vdash ?A, ?A, B, \Gamma}{\vdash ?A, B, \Gamma}\ c?}{\vdash ?A\sharp B, \Gamma, ?A}\ \sharp \qquad \longleftrightarrow \qquad \dfrac{\dfrac{\vdash ?A, ?A, B, \Gamma}{\vdash ?A, ?A\sharp B, \Gamma, ?A}\ \sharp}{\vdash ?A\sharp B, \Gamma, ?A}\ c?$$

According to the standard approach, the rules $c?$ and $\sharp$ can be permuted even though they are not in a GP permutation position in the left subproof. The reason for this is that the formula $?A$ is at the same time the active and principal formula of the $c?$ rule and an active formula of the $\sharp$ rule. Thus, in the left subproof above, the rule $\sharp$ consumes formula $?A$ which is a $quasi$ active formula of the $c?$ rule and, as such, it is present in the premise and available for the rule $\sharp$, after eventual inversion of the rules. Also, a very important detail here is the "restoration" of formula $?A$ through the principal part of the $\sharp$ rule. So, after inversion, $?A$ is available again for the $c?$ rule. Let us now concentrate on the above 'restoration'. Consider the current definition of a $principal$ formula. Note that among principal formulae of the rule $\sharp$ there are formulae (denoted by $?\delta$ in the above schema-rule) with no sub-formulae appearing in the premise(s).

Such formulae are distinguished from those principal formulae that are the result of the conversion of certain active formulae (*i.e.* that have some sub-formulae in the premise(s)). These formulae are referred to as $extra$ formulae. An $extra$ formula can be used in a permutation (as shown in the previous example), to 'restore' some formula(e) which are already consumed.

2.3. A Refined Structure of Sequent Rules

On the basis of examination of sequent rule schemas proposed by the authors in [16] a more refined syntax for sequent calculus rules is specified by the following definition. Many well known sequent calculi fit into this framework. Among them are various sequent calculus formalizations of classical logic, liner logic, intuitionistic logic, affine and modal logic.

The application of an inference rule is described by identifying, for each premise, the multisets of formulae:

- $\mathcal{A}$: formulae whose presence is required for the inference but which do not appear in the conclusion;

- $\mathcal{QA}$: formulae whose presence is required for the inference but which continue to appear in the conclusion;

- $Context$: formulae whose presence is not required for the inference but are copied into the conclusion.

The conclusion is also divided into multisets:

- $\mathcal{P}$: formulae that are newly inferred by this inference rule;

- $\mathcal{E}$: extra formulae in the conclusion that do not arise from any premise (*i.e.* that can be instantiated independently of premises of inference rule);

- $\mathcal{QA}, Context$: coming from $\mathcal{QA}/Context$ of premise(s).

Intuitively, a rule should contain at most one principal formula. However, there are some sequent systems in which this is not the case. For example, the following rule $\multimap L$ from a specialized fragment of ILL [9] has more then one principal formula.

$$\frac{\Gamma \vdash \phi}{\Gamma, \phi \multimap A \vdash A}\ \multimap L$$

As allowing rules of this form complicates the analysis, in this paper we restrict our attention to rules in which there is at most one principal formula[6].

[6] Note that the w rule (in Table 1 below) has no principal formula.
In terms of automation of rules a general way to cover cases like the rule $\multimap L$ and to syntactically "detect" all principal formulae would be to identify all *inferred formulae* of rule I. An *inferred metaformula* of rule I is either a logical constant or a metaformula which occurs in the conclusion of rule I but does not exist in the premise(s). Let the multiset of all inferred metaformulae of I be Inf. A *principal metaformula* of I is a single-formula metavariable in Inf, where at least one of its submetaformulae occurs in the premise(s). Let the multiset of all principal metaformulae identified in

Definition 1. *(The structure of a sequent calculus rule I)*

- *A* principal metaformula *of rule I is either a logical constant which occurs in the conclusion of rule I but does not exist in the premise(s) or a single-formula metavariable which occurs in the conclusion of rule I but does not exist in the premise(s), where at least one of its submetaformulae occurs in the premise(s).*
- *An* extra metaformula *of rule I is a metaformula which occurs in the conclusion of rule I but not in the premise(s), where none if its submetaformulae occurs in the premise(s).*
- *An* active metaformula *of rule I is a metaformula which occurs in a premise but does not exist in the conclusion.*
- *A* quasi-active metaformula *of a rule I is a metaformula whose presence in a premise is necessary (but not sufficient) for the rule to be applied, and which is copied unchanged into the conclusion of the rule.*
- *A* context metaformula *of a rule I is a metaformula that is neither active, quasi-active, principal nor extra.*

For an inference rule I, the active, quasi-active, principal, extra and context formulae are a result of instantiation of the corresponding metaformulae of the rule I.

The active part *(denoted by $\mathcal{A}_I^i$) and the* quasi-active part *(denoted by $\mathcal{QA}_I^i$) of the i-th premise of an inference I are the (possibly empty) multisets of its active and quasi-active formulae, respectively. The* principal part *(denoted by $\mathcal{P}_I$) and the* extra part *of an inference I (denoted by $\mathcal{E}_I$) are the (possibly empty) multisets of its principal and extra formulae, respectively. The* context *of the i-th premise of an inference I (denoted by $Context_I^i$) is the (possibly empty) multiset - complement of its active and quasi-active parts.*

The *active, quasi-active* and *context* parts of the i-th premise of an inference I are decorated with superscript i: $\mathcal{A}_I^i$, $\mathcal{QA}_I^i$, $Context_I^i$. The value $i = 1$ (or $i = 2$) corresponds to the left (or right) premise of I. In the case of a unary rule I, superscript i will be omitted. Also, in the case of a binary rule I, the *active, quasi-active* and *context* parts will be denoted by $\mathcal{A}_I = \mathcal{A}_I^1, \mathcal{A}_I^2$, $\mathcal{QA}_I = \mathcal{QA}_I^1, \mathcal{QA}_I^2$ and $Context_I = Context_I^1, Context_I^2$, respectively.
$\mathcal{A}_I^{'}, q\mathcal{A}_I^{'}, \mathcal{C}_I^{'}, E_I^{'}, \mathcal{A}_I^{''}, q\mathcal{A}_I^{''} \ldots$ will also be used to denote (possibly empty) multisets of active, quasi-active, context and extra formulae, respectively.

The above definition is illustrated with some examples in Table 1 below.

Note that the metaformula F in the rule from LJT is a context metaformula rather than quasi-active, as in this system the constraint on the succedents is that there is at most one formula. However, if a similar restriction to requiring *exactly* one formula is made, then F becomes quasi-active, rather than context.

Inf be $Inf_{\mathcal{P}}$. A single-formula metavariable in $Inf \setminus Inf_{\mathcal{P}}$, where at least one of its submetaformulae occurs in $Inf_{\mathcal{P}}$ is deemed to be a principal formula too. An *extra metaformula* of I is an inferred metaformula which is not a principal formula. However this approach is an item of future work.

In this way we can think of context formulae as an indication of the amount of freedom specified by the rule.

Table 1.

Logic	Rule	$\mathcal{A}$	$\mathcal{QA}$	$\mathcal{P}$	$\mathcal{E}$	$Context$
CLL [28]	$\dfrac{\Gamma \vdash \Delta}{\Gamma \vdash ?A, \Delta}\ w$				$?A$	Γ, Δ
CLL	$\dfrac{\Gamma \vdash ?A, ?A, \Delta}{\Gamma \vdash ?A, \Delta}\ c?$	$?A$	$?A$			Γ, Δ
LM [29]	$\dfrac{A, \Gamma \vdash B}{\Gamma \vdash A \supset B, \Delta}\ \supset_R$	A, B		$A \supset B$	Δ	Γ
CLL	$\dfrac{\Gamma \vdash A, \Delta \quad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ Cut$	$A,\ A$				$\Gamma, \Delta, \Gamma', \Delta'$
CLL	$\dfrac{}{\Gamma \vdash \top, \Delta}\ \top$			$\top$	Γ, Δ	
CLL	$\dfrac{\Gamma \vdash \Delta}{\Gamma \vdash \bot, \Delta}\ \bot$			$\bot$		Γ, Δ
LJT [6]	$\dfrac{B, a, \Gamma \vdash F}{a \supset B, a, \Gamma \vdash F}\ \supset_1 L$	B	a	$a \supset B$		Γ, F
LJT $multip.$ $-concl.$ [6]	$\dfrac{A \supset B, \Gamma \vdash A, \Delta \quad B, \Gamma \vdash \Delta}{A \supset B, \Gamma \vdash \Delta}\ \supset L$	A, B	$A \supset B$			$\Gamma, \Delta, \Gamma, \Delta$
S4 [29]	$\dfrac{\Box\Gamma, \Gamma' \vdash \Box A, \Diamond\Delta, \Delta'}{\Box\Gamma \vdash A, \Diamond\Delta}\ \Box R$	A		$\Box A$	Γ', Δ'	$\Box\Gamma, \Box\Delta$
S4	$\dfrac{\Gamma \vdash \Delta, \Diamond A, A}{\Gamma \vdash \Delta, \Diamond A}\ \Diamond R$	A	$\Diamond A$			Γ, Δ
S4	$\dfrac{\Gamma \vdash \Delta, A \wedge B, A \quad \Gamma \vdash \Delta, A \wedge B, B}{\Gamma \vdash \Delta, A \wedge B}\ \wedge R$	A, B	$A \wedge B, A \wedge B$			Γ, Δ
Lax $logic$ [7]	$\dfrac{\Gamma, \bigcirc A, A \vdash \bigcirc B}{\Gamma, \bigcirc A \vdash \bigcirc B}\ \bigcirc L$	A	$\bigcirc A, \bigcirc B$			Γ

Binary rules are categorised according to how the contexts of the two rules are combined. A distinction is made between *multiplicative* rules, where the context of each premise is copied unchanged into the conclusion (for example the cut rule), and *strong additive* rules, where the context of each premise and of the conclusion are identical (for example the rule $\&R$ from linear logic (see Figure 1)). Note that there are also *weak additive* rules where some parts of the context of the premise and of the conclusion are identical. For reasons of simplicity, the reminder of the paper will focus on one-sided sequent calculus rules. Generally, they have the following form for the unary, multiplicative and additive binary rules, respectively:

$$\frac{\vdash \mathcal{A}_I, \quad \mathcal{QA}_I, \quad Context_I}{\vdash \mathcal{P}_I, \quad \mathcal{QA}_I, \quad Context_I, \quad \mathcal{E}_I}\ I$$

$$\frac{\vdash \mathcal{A}_I^1, \quad \mathcal{QA}_I^1, \quad Context_I^1 \qquad \vdash \mathcal{A}_I^2, \quad \mathcal{QA}_I^2, \quad Context_I^2}{\vdash \mathcal{P}_I, \quad \mathcal{QA}_I^1, \mathcal{QA}_I^2, \quad Context_I^1, Context_I^2, \quad \mathcal{E}_I}\ I$$

$$\frac{\vdash \mathcal{A}_I^1, \quad \mathcal{QA}_I^1, \quad Context_I \qquad \vdash \mathcal{A}_I^2, \quad \mathcal{QA}_I^2, \quad Context_I}{\vdash \mathcal{P}_I, \quad \mathcal{QA}_I^1, \mathcal{QA}_I^2, \quad Context_I, \quad \mathcal{E}_I}\ I$$

The proposed representation of sequent calculi rules improves on prior approaches to characterising permutability. The remainder of this subsection contains a brief overview of the authors' previous work ([16]) in this direction.

The notion *being in a strong permutation position* is introduced to cover some situations which cannot be detected by the *GP permutation position*. Informally, two adjacent inferences I and J in a given derivation $\begin{array}{c}[\mathcal{D}]\\ I\\ J\end{array}$ are in the *strong permutation position* iff they verify the following conditions:

i) The principal formula of the upper inference I is not an active or quasi-active formula of the lower inference J
ii) If I is a binary rule, then the corresponding active and quasi-active formulae of J belong to the same premise of I
iii) A formula which is a quasi-active formula of the upper inference I and an active formula for the lower inference J, there is a separate occurrence in the extra part of J.

The condition ii) above takes into account more carefully the distribution of formulae from $\mathcal{A}_J^p, \mathcal{QA}_J^p$ between the different premises (if any) of I. Such a situation will not be "recognised" in the GP permutation position:

$$\dfrac{\dfrac{\vdash a, a^\perp \qquad \vdash b, b^\perp}{\vdash a \otimes b, a^\perp, b^\perp}\ \otimes}{\vdash a^\perp \wp b^\perp, a \otimes b}\ \wp \qquad\qquad \dfrac{\dfrac{\vdash a, a^\perp \qquad \dfrac{\vdash b, b^\perp}{\vdash ?c, b, b^\perp}\ w}{\vdash a \otimes b, a^\perp, ?c, b^\perp}\ \otimes}{\vdash b^\perp \wp ?c, a \otimes b, a^\perp}\ \wp$$

not permutable rules	permutable rules
GP ($\not\Rightarrow$ strong) permutation pos.	strong ($\Rightarrow$ GP) permutation pos.

The condition iii) above ensures that a quasi-active formula(es) of the upper rule I which is(are) "consumed"[7] by the lower J is(are) restored through the extra part of J. In this way the formula(e) remains available for I, after the eventual reversing of the order of I and J. As pointed out in Example 2 of Section 2.2, such a situation cannot be detected and recognised by the condition $\boldsymbol{A}$.

The notion "being in a strong permutation position" is made more precise in the following definition. The intention was to specify conditions as constraints (as far as possible) between the multisets of the active, quasi-active, context and extra formulae.

Thus, for any two adjacent rules I and J, where I is above the p-th premise of J ($p \in \{1, 2\}$), the conditions i) and ii) are specified as $\mathcal{A}_J^p, \mathcal{QA}_J^p \subseteq Context_I^k, \mathcal{QA}_I^k$ and $(\mathcal{A}_J^p \not\subseteq Context_I^k)$, while the condition iii) is equivalent with the constraint $(\mathcal{A}_J^p \not\subseteq Context_I^k) \Rightarrow (\mathcal{A}_J^p \backslash Context_I^k \subseteq \mathcal{E}_J)$.

[7] *i.e.* used as the active formulae(e) for J.

Definition 2. *(strong permutation position)*
Two adjacent inferences I and J of a given derivation $\begin{array}{c}[\mathcal{D}]\\ I\\ J\end{array}$ are in the *strong permutation position* iff:
If I is a unary rule, then

$\mathcal{A}^p_J, \mathcal{QA}^p_J \subseteq Context_I, \mathcal{QA}_I$ and $(\mathcal{A}^p_J \not\subseteq Context_I) \Rightarrow (\mathcal{A}^p_J \backslash Context_I \subseteq \mathcal{E}_J)$,

else for a binary rule $I \quad \exists k \in \{1, 2\}$

$\mathcal{A}^p_J, \mathcal{QA}^p_J \subseteq Context^k_I, \mathcal{QA}^k_I$ and $(\mathcal{A}^p_J \not\subseteq Context^k_I) \Rightarrow (\mathcal{A}^p_J \backslash Context^k_I \subseteq \mathcal{E}_J)$.

The value $k = 1$ (or $k = 2$) corresponds to the *strong-left* (or *strong-right*) permutation position.

Note that the notion of *the strong permutation position* is a refinement of the *GP permutation position* and is also a necessary condition for checking the permutations of the inference rules in a given derivation. In order to study possible permutations, the following definition is adopted:

Definition 3. *(inference permutability)*
An inference I is permutable over an inference J in a given derivation $\begin{array}{c}[\mathcal{D}]\\ I\\ J\end{array}$ iff the inferences satisfy the conditions:

$\boldsymbol{A^s}$. I and J are in a strong permutation position;

$\boldsymbol{B_1}$. J is applicable on the appropriate premise(s) of I;

$\boldsymbol{B_2}$. The end-sequent of $\begin{array}{c}[\mathcal{D}]\\ J\end{array}$ can be a premise of I;

$\boldsymbol{C}$. Derivations $\begin{array}{c}[\mathcal{D}]\\ I\\ J\end{array}$ and $\begin{array}{c}[\mathcal{D}]\\ J\\ I\end{array}$ have the same conclusion and the same set of leaves (modulo renaming certain free variables).

In all other cases, it is assumed that I is *not permutable over* J.

The condition $\boldsymbol{C}$ implies that permutations do not alter the constituent of sub-branches and that a sub-branch may be deleted only if it is duplicated. In particular, a permutation preserves the set of premises of a given derivation. Note that permutations which arbitrarily rewrite a sub-branch of a derivation are not allowed. For example, as the $\top$ rule can arbitrarily rewrite a sub-branch of a derivation, the following transformation (page 4 of [4]) is not a permutation (the same approach is used in [4]):

$$\dfrac{\begin{array}{c}\vdots\ \mathcal{D}\\ \vdash A, \Gamma_1\end{array} \qquad \dfrac{}{\vdash B, \Gamma_2, \top}\,\top}{\vdash A \otimes B, \Gamma_1, \Gamma_2, \top}\,\otimes \qquad \longmapsto \qquad \dfrac{}{\vdash A \otimes B, \Gamma_1, \Gamma_2, \top}\,\top$$

However, as we shall see shortly, the condition $\boldsymbol{C}$ is a bit stronger than one in fact needs.

The proposed characterisation of the structure of sequent calculi rules is used for the *automated-oriented specification* of the permutation process. The proposed approach is illustrated (in [16], Theorems 4.4, 4.5 and 4.6) using an example of the permutation of a unary rule over a strong additive rule. Some permutation "pre-conditions" may be analyzed in detail and restricted to particular parts of the inference rules, with minimally sufficient tests. In this paper, using the same technique, some special cases of permutations are recognized and generalized and a number of certain subtler proof rearrangements specified when permutations are not strictly possible.

3. Further Applications of the Refined Sequent Structure

In this section the authors of this paper extend previous work [16] to include broader aspects of permutation analysis.

3.1. General Permutations

The proposed classification of the formulae of sequent rules makes it possible to note a number of certain connections between the structure of the rules and the strong permutation position, as specified in the Lemma below. Thus, for example, by analyzing the linear logic rule $\dfrac{\vdash \Gamma}{\vdash \Gamma, ?\phi}\ w$ independently of the context in which it is used in a proof, a useful permutability property can be extracted: the rule w is *always in the strong permutation position*[8] with the rule that precedes it in a proof.

Lemma 1.
1) A rule which satisfies the following condition: $\mathcal{A} = \mathcal{QA} = \mathcal{P} = \emptyset \ \wedge \ \mathcal{E} \neq \emptyset \ \wedge \ Context \neq \emptyset$ *is always in a strong permutation position with the rule that precedes it in a proof.*
2) A rule which satisfies the following condition: $\mathcal{QA} = \mathcal{P} = \mathcal{E} = \emptyset \ \wedge \ \mathcal{A} \neq \emptyset \ \wedge \ Context \neq \emptyset$ *is always in a strong permutation position with the rule that follows it in a proof.*
3) A binary rule which satisfies $\mathcal{A} = \mathcal{QA} = \mathcal{P} = \mathcal{E} = \emptyset \ \wedge \ Context \neq \emptyset$ *is always in a strong permutation position with the rule that is above it as well as with the rule that is below it in a subproof.*
Proof: Straightforward from Definitions 2 and 3. □

Let us illustrate the "groups" of rules specified in Lemma 1 with some examples from the standard sequent calculus for linear logic and a variation of it in which implicit weakening is used [27].
In examples below A and B denote arbitrary formulae, $A_?$ denotes a ?-prefixed formula or a constant $\bot$, $?\Gamma$ denotes that all formulae in multiset Γ are prefixed with ?-labels, $\Gamma_l = \Delta, ?\Sigma, ?\Theta_l$, $\Gamma_r = \Delta, ?\Sigma, ?\Theta_r$, and $\Gamma_{lr} = \Delta, ?\Sigma, ?\Theta_l, ?\Theta_r$ where $?\Theta_l \cap ?\Theta_r = \emptyset$.
1)

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?\phi}\ w \qquad \frac{\vdash \Gamma \quad \vdash \Delta}{\vdash \Gamma, \Delta, A_? \otimes B_?}\ \otimes_{lr}{}^{9} \qquad \frac{\vdash \Gamma_l \quad \vdash \Gamma_r}{\vdash \Gamma_{lr}, A_? \& B_?}\ \&_{lr}{}^{9} \qquad \frac{\vdash ?\Gamma}{\vdash ?\Gamma, !A}\ ! - \text{storage}^{9}$$

[8] Even stronger: always permutable over the rule that precedes it in a proof.
[9] From one-sided, implicit weakening fragment of linear logic [27].

2) $\dfrac{\vdash A, \Gamma \quad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}$ Cut 3) $\dfrac{\vdash \Gamma \quad \vdash \Delta}{\vdash \Gamma, \Delta}$ mix

The initial steps in the automation of the permutation process include the specification of the conditions of Definition 3. The very first step is a detailed specification of all changes which may occur during a permutation. The following proposition extends the result of Proposition 4.3 (of [16]) to deal with arbitrary rules I and J.

Proposition 1.
1) The active, quasi-active, principal and extra parts of an inference rule are not changed by its permutation up or down, in a given proof.
2) For two adjacent rules I and J, being in strong permutation position, where I is above the p-th premise of J ($p \in \{1,2\}$), the context parts of permuted rules I and J are changed after the permutation as follows:

$$Context_I^k := \begin{cases} [Context_I^k, \mathcal{E}_J] \backslash \mathcal{A}_J\ ,\ \mathcal{P}_J, & \text{if } J \text{ is a unary rule} \\ [Context_I^k, \mathcal{E}_J] \backslash \mathcal{A}_J^p\ ,\ \mathcal{P}_J\ ,\ \mathcal{QA}_J^{\bar{p}}\ ,\ \mathcal{G}_J, & \text{if } J \text{ is a binary rule} \end{cases}$$

$$Context_J^p := \begin{cases} Context_J^p \backslash [\mathcal{P}_I, \mathcal{E}_I]\ ,\ \mathcal{A}_I, & \text{if } I \text{ is a unary rule} \\ Context_J^p \backslash [\mathcal{P}_I, \mathcal{E}_I, \mathcal{QA}_I^{\bar{k}}, \mathcal{G}_I]\ ,\ \mathcal{A}_I^k, & \text{if } I \text{ is a binary rule} \end{cases}$$

where the value $k = 1$ ($k = 2$) corresponds to the *strong-left* (*strong-right*) permutation position, $\bar{k} = 3 - k$, $\bar{p} = 3 - p$, and multisets $\mathcal{G}_I$ and $\mathcal{G}_J$ are defined as follows:

$$\mathcal{G}_I = \begin{cases} \emptyset, & \text{if } I \text{ is a strong-additive rule} \\ Context_I^{\bar{k}}, & \text{otherwise} \end{cases} \qquad \mathcal{G}_J = \begin{cases} \emptyset, & \text{if } J \text{ is a strong-additive rule} \\ Context_J^{\bar{p}}, & \text{otherwise} \end{cases}$$

Proof: The proof is based on a simple inspection of the distribution of formulae before and after reversing the order of the rules in question. For example, in the case of a multiplicative binary rule I and strong additive binary rule J (assuming that I is above the left premise (*i.e.* $p = 1$) of J and the strong-left permutation position (*i.e.* $k = 1$)), the starting subproof and the result of permutation would be as follows, respectively:

$$\dfrac{\dfrac{\vdash \mathcal{A}_I^1,\ \underbrace{\mathcal{QA}_I^1}_{\mathcal{A}'_J, \mathcal{QA}'_J, q\mathcal{A}'_I}\ ,\ \underbrace{Context_I^1}_{\mathcal{A}''_J, \mathcal{QA}''_J, \mathcal{C}''_I} \qquad \vdash \mathcal{A}_I^2,\ \mathcal{QA}_I^2,\ Context_I^2}{\vdash \underbrace{\mathcal{P}_I,\ \mathcal{QA}_I^1, \mathcal{QA}_I^2,\ Context_I^1, Context_I^2,\ \mathcal{E}_I}}\ I \qquad}{}$$

$$\dfrac{\vdash \underbrace{\mathcal{A}'_J, \mathcal{A}''_J}_{\mathcal{A}_J^1},\ \underbrace{\mathcal{QA}'_J, \mathcal{QA}''_J}_{\mathcal{QA}_J^1},\ \underbrace{\mathcal{P}_I, \mathcal{QA}_I^2, q\mathcal{A}'_I, \mathcal{C}''_I, Context_I^2, \mathcal{E}_I}_{Context_J} \qquad \vdash \mathcal{A}_J^2, \mathcal{QA}_j^2, Context_J}{\vdash \mathcal{P}_J,\ \mathcal{QA}_J^1, \mathcal{QA}_J^2,\ Context_J,\ \underbrace{\mathcal{E}_J}_{\mathcal{E}'_J, \mathcal{A}'_J}}\ J$$

$$
\dfrac{\underbrace{\dfrac{\underbrace{\vdash \mathcal{A}^1_I,\ \underbrace{\mathcal{QA}^1_I}_{\mathcal{A}'_J,\mathcal{QA}'_J,q\mathcal{A}'_I},\ \underbrace{Context^1_I}_{\mathcal{A}''_J,\mathcal{QA}''_J,\mathcal{C}''_I}}_{\vdash \underbrace{\mathcal{A}'_J,\mathcal{A}''_J}_{\mathcal{A}^1_J},\ \underbrace{\mathcal{QA}'_J,\mathcal{QA}''_J}_{\mathcal{QA}^1_J},\ \underbrace{\mathcal{A}^1_I,q\mathcal{A}'_I,\mathcal{C}''_I}_{Context_J}} \qquad \vdash \mathcal{A}^2_J,\mathcal{QA}^2_j,Context_J}{\vdash \mathcal{P}_J,\mathcal{QA}^1_J,\mathcal{QA}^2_J,Context_J,\ \underbrace{\mathcal{E}_J}_{\mathcal{E}'_J,\mathcal{A}'_J}}\,J}_{\vdash \mathcal{A}^1_I,\ \underbrace{\mathcal{QA}'_J,\mathcal{A}'_J,q\mathcal{A}'_I}_{\mathcal{QA}^1_I},\ \mathcal{P}_J,\mathcal{E}'_J,\mathcal{C}''_I,\mathcal{QA}''_J,\mathcal{QA}^2_J} \qquad \vdash \mathcal{A}^2_I,\mathcal{QA}^2_I,Context^2_I}{\vdash \mathcal{P}_I,\ \mathcal{QA}^1_I,\mathcal{QA}^2_I,\ \mathcal{P}_J,\mathcal{E}'_J,\mathcal{C}''_I,\mathcal{QA}''_J,\mathcal{QA}^2_J,Context^2_I,\ \mathcal{E}_I}\,I
$$

Upon comparison of the conclusions of rules I and J before and after the permutation, it follows that, after the permutation, the $Context$ parts are changed as follows: $Context^1_I := [Context^1_I, \mathcal{E}_J] \backslash \mathcal{A}^1_J\ ,\ \mathcal{P}_J\ ,\ \mathcal{QA}^2_J\ ,\ \mathcal{G}_J$
$Context_J := Context_J \backslash [\mathcal{P}_I, \mathcal{E}_I, \mathcal{QA}^2_I, \mathcal{G}_I]\ ,\ \mathcal{A}^1_I$

□

One of the objectives is to minimize the amount of testing done in the permutation process. Since according to Proposition 1 the $context$ part is the only part of a rule which can change during a permutation, it is sufficient to concentrate the conditions $\boldsymbol{B_1}$ and $\boldsymbol{B_2}$ on the $context$ parts of the given rules. It is enough to check whether or not the multisets, specified in Proposition 1, can be accepted as (*i.e.* can be instantiated in) the contexts of the corresponding rules after the eventual permutation. Also, analysis shows that, in some cases, the condition $\boldsymbol{C}$ of Definition 3 is fulfilled automatically after the reversal of the rules, and therefore the checking of this condition can be omitted in such cases.

Using the same technique as in [16], it is straightforward to specify all the necessary and sufficient conditions for a permutation of two arbitrary consecutive rules. To simplify the presentation, in the rest of the paper we will assume that for any two adjacent inferences I and J (whenever J is a binary rule and directly follows I), I is above the left (*i.e.* first) premise $\vdash \mathcal{A}^1_J, \mathcal{QA}^1_J, Context^1_J$ of J (unless explicitly stated otherwise). Our results can be easily generalized to deal with the cases where an inference I is above the $p-th$ premise of an inference J.

Let $\alpha, \beta, \gamma, \alpha_1, \beta_1, \gamma_1, \ldots$ denote sequents, $\mathcal{ID}$ the initial derivation[10] and $\mathcal{PR}_{l.k}\ \ l \in \{1,2,3,4\}, k \in \{1,2,3\}$[11] the possible forms of permutation results.

[10] Consistent with the previous assumption that I is above the first (*i.e.* left) premise of J in the starting derivation.

[11] The value $k = 1$ (respectively $k = 2$) corresponds to the *strong-left* (resp. *strong-right*) permutation position. The value $k = 3$ corresponds to the situation when the strong-left and strong-right permutation positions apply simultaneously.

The *permutation result* $\mathcal{PR}_{l.k}$ for two adjacent rules I and J can have different forms in regard to possible duplication of certain proof branches or duplication of some rules in question. Figure 2 shows possible permutations of given inference rules I and J. Note that I and J can be either additive or multiplicative here.

Fig. 2. Possible permutations of rules I and J

Case 1: I and J unary rules

$$\mathcal{ID}\quad \cfrac{\cfrac{\begin{array}{c}\vdots\ \mathcal{D}\\ \gamma\end{array}}{\beta}\,I}{\alpha}\,J \quad\longmapsto\quad \mathcal{PR}_{1.1}\quad \cfrac{\cfrac{\begin{array}{c}\vdots\ \mathcal{D}\\ \gamma\end{array}}{\beta_1}\,J}{\alpha}\,I$$

Case 2: I unary, J binary rule

$$\mathcal{ID}\quad \cfrac{\cfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \gamma\end{array}}{\beta_1}\,I \qquad \begin{array}{c}\vdots\ \mathcal{D}_2\\ \beta_2\end{array}}{\alpha}\,J \quad\longmapsto\quad \mathcal{PR}_{2.1}\quad \cfrac{\cfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \gamma\end{array} \qquad \begin{array}{c}\vdots\ \mathcal{D}_2\\ \beta_2\end{array}}{\beta_1'}\,J}{\alpha}\,I$$

Case 3: I binary, J unary rule

$$\mathcal{ID}\quad \cfrac{\cfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \gamma_1\end{array} \qquad \begin{array}{c}\vdots\ \mathcal{D}_2\\ \gamma_2\end{array}}{\beta}\,I}{\alpha}\,J \quad\longmapsto\quad \begin{cases} \mathcal{PR}_{3.1}\quad \cfrac{\cfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \gamma_1\end{array}}{\beta_1}\,J \qquad \begin{array}{c}\vdots\ \mathcal{D}_2\\ \gamma_2\end{array}}{\alpha}\,I \\[2ex] \mathcal{PR}_{3.2}\quad \cfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \gamma_1\end{array} \qquad \cfrac{\begin{array}{c}\vdots\ \mathcal{D}_2\\ \gamma_2\end{array}}{\beta_2}\,J}{\alpha}\,I \\[2ex] \mathcal{PR}_{3.3}\quad \cfrac{\cfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \gamma_1\end{array}}{\beta_1}\,J \qquad \cfrac{\begin{array}{c}\vdots\ \mathcal{D}_2\\ \gamma_2\end{array}}{\beta_2}\,J}{\alpha}\,I \end{cases}$$

Case 4: I and J binary rules

$$\mathcal{ID}\quad \cfrac{\cfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \gamma_1\end{array} \qquad \begin{array}{c}\vdots\ \mathcal{D}_2\\ \gamma_2\end{array}}{\beta_1}\,I \qquad \begin{array}{c}\vdots\ \mathcal{D}_3\\ \beta_2\end{array}}{\alpha}\,J \quad\longmapsto\quad \begin{cases} \mathcal{PR}_{4.1}\quad \cfrac{\cfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \gamma_1\end{array} \qquad \begin{array}{c}\vdots\ \mathcal{D}_3\\ \beta_2\end{array}}{\delta_1}\,J \qquad \begin{array}{c}\vdots\ \mathcal{D}_2\\ \gamma_2\end{array}}{\alpha}\,I \\[2ex] \mathcal{PR}_{4.2}\quad \cfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \gamma_1\end{array} \qquad \cfrac{\begin{array}{c}\vdots\ \mathcal{D}_2\\ \gamma_2\end{array} \qquad \begin{array}{c}\vdots\ \mathcal{D}_3\\ \beta_2\end{array}}{\delta_2}\,J}{\alpha}\,I \\[2ex] \mathcal{PR}_{4.3}\quad \cfrac{\cfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \gamma_1\end{array} \qquad \begin{array}{c}\vdots\ \mathcal{D}_3\\ \beta_2\end{array}}{\delta_1}\,J \qquad \cfrac{\begin{array}{c}\vdots\ \mathcal{D}_2\\ \gamma_2\end{array} \qquad \begin{array}{c}\vdots\ \mathcal{D}_3\\ \beta_2\end{array}}{\delta_2}\,J}{\alpha}\,I \end{cases}$$

Consider the permutation of two binary rules I and J. The following is an illustration of a formal specification of the sufficient and necessary conditions for the permutation of a multiplicative rule over a strong additive rule:

Theorem 1. *Given a derivation* $\mathcal{ID}$ $\cfrac{\cfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ \gamma_1\end{array} \qquad \begin{array}{c}\vdots\ \mathcal{D}_2\\ \gamma_2\end{array}}{\beta_1}\,I \qquad \begin{array}{c}\vdots\ \mathcal{D}_3\\ \beta_2\end{array}}{\alpha}\,J$.

1) A multiplicative rule I is permutable over a strong additive rule J in derivation $\mathcal{ID}$ yielding a permutation result of the form $\mathcal{PR}_{4.k},\ \ k \in \{1,2\}$ iff the inferences I and J satisfy the following conditions in derivation $\mathcal{ID}$:

i) I *and* J *are in strong permutation position;*
ii) $\mathcal{P}_I = \emptyset, \quad \mathcal{E}_I, \mathcal{QA}_I^{\overline{k}}, Context_I^{\overline{k}} = \mathcal{A}_I^k$*;*
iii) [12] *The multiset* $[Context_I^k, \mathcal{E}_J] \backslash \mathcal{A}_J^1, \quad \mathcal{P}_J, \ \mathcal{QA}_J^2$ *can be (i.e. satisfies all restrictions for) the context* $Context_I^k$ *of rule* I*.*

2) A multiplicative rule I *is permutable over a strong additive rule* J *in derivation* $\mathcal{ID}$ *yielding a permutation result of the form* $\mathcal{PR}_{4.3}$ *iff the inferences* I *and* J *satisfy the following conditions in derivation* $\mathcal{ID}$*:*

i) I *and* J *are in the strong-left and strong-right permutation positions;*
ii) $\forall k \in \{1,2\} \quad \mathcal{P}_I = \emptyset, \quad \mathcal{E}_I, \ \mathcal{QA}_I^{\overline{k}}, \ Context_I^{\overline{k}} = \mathcal{A}_I^k$*;*
$\mathcal{P}_J = \emptyset, \quad \mathcal{E}_J, \ \mathcal{QA}_J^2 = \mathcal{A}_J^1$*;*
iii) $\forall k \in \{1,2\}$ *The multiset* $[Context_I^k, \mathcal{E}_J] \backslash \mathcal{A}_J^1, \quad \mathcal{P}_J, \ \mathcal{QA}_J^2$ *can be (i.e. satisfies all restrictions for) the context* $Context_I^k$ *of rule* I*.*

Proof:
1) $\boxed{\Rightarrow:}$ As I is permutable over J, condition i) follows directly from Definition 3. According to Proposition 1, after the permutation, the new context in the left premise of J is the multiset $Context_J \backslash [\mathcal{P}_I, \mathcal{E}_I, \mathcal{QA}_I^{\overline{k}}, Context_I^{\overline{k}}]$, $\mathcal{A}_I^k$. As J is a strong additive rule (and the context of the premises must be identical), it must be that: $Context_J = Context_J \backslash [\mathcal{P}_I, \mathcal{E}_I, \mathcal{QA}_I^{\overline{k}}, Context_I^{\overline{k}}]$, $\mathcal{A}_I^k$

$$\iff \quad \mathcal{P}_I, \mathcal{E}_I, \mathcal{QA}_I^{\overline{k}}, Context_I^{\overline{k}} = \mathcal{A}_I^k.$$

As (for any rule) $\mathcal{P}_I \cap \mathcal{A}_I^k = \emptyset$, it follows that $\mathcal{P}_I = \emptyset$ and $\mathcal{E}_I, \mathcal{QA}_I^{\overline{k}}, Context_I^{\overline{k}} = \mathcal{A}_I^k$ *i.e.* the condition ii) is fulfilled.

According to Proposition 1, after the permutation, the new context of the corresponding premise of I is the multiset $[Context_I^k, \mathcal{E}_J] \backslash \mathcal{A}_J^1, \quad \mathcal{P}_J, \ \mathcal{QA}_J^2$ which implies condition iii).

$\boxed{\Leftarrow:}$ Without loss of generality, we assume that I and J are in the strong-left (*i.e.* $k = 1$) permutation position. Then we have the following distribution of formulae in the starting $\mathcal{ID}$ derivation:

$$\dfrac{\dfrac{\vdash \mathcal{A}_I^1, \quad \underbrace{\mathcal{QA}_I^1}_{\mathcal{A}'_J, \mathcal{QA}'_J, q\mathcal{A}'_I} \ , \ \underbrace{Context_I^1}_{\mathcal{A}''_J, \mathcal{QA}''_J, \mathcal{C}''_I} \qquad \vdash \mathcal{A}_I^2, \ \mathcal{QA}_I^2, \ Context_I^2}{\vdash \underbrace{\mathcal{P}_I, \ \mathcal{QA}_I^1, \mathcal{QA}_I^2, \ Context_I^1, Context_I^2, \ \mathcal{E}_I}_{}}\,I}{\vdash \underbrace{\mathcal{A}'_J, \mathcal{A}''_J}_{\mathcal{A}_J^1}, \ \underbrace{\mathcal{QA}'_J, \mathcal{QA}''_J}_{\mathcal{QA}_J^1}, \ \underbrace{\mathcal{P}_I, \mathcal{QA}_I^2, q\mathcal{A}'_I, \mathcal{C}''_I, Context_I^2, \mathcal{E}_I}_{Context_J} \qquad \vdash \mathcal{A}_J^2, \mathcal{QA}_j^2, Context_J}$$

$$\dfrac{\vdash \underbrace{\mathcal{A}'_J, \mathcal{A}''_J}_{\mathcal{A}_J^1}, \ \underbrace{\mathcal{QA}'_J, \mathcal{QA}''_J}_{\mathcal{QA}_J^1}, \ \underbrace{\mathcal{P}_I, \mathcal{QA}_I^2, q\mathcal{A}'_I, \mathcal{C}''_I, Context_I^2, \mathcal{E}_I}_{Context_J} \qquad \vdash \mathcal{A}_J^2, \mathcal{QA}_j^2, Context_J}{\vdash \mathcal{P}_J, \ \mathcal{QA}_J^1, \mathcal{QA}_J^2, \ Context_J, \ \underbrace{\mathcal{E}_J}_{\mathcal{E}'_J, \mathcal{A}'_J}}\,J$$

[12] This condition can be generalized to deal with the cases where an inference I is above the p-th premise of J, as follows: The multiset $[Context_I^k, \mathcal{E}_J] \backslash \mathcal{A}_J^p, \quad \mathcal{P}_J, \ \mathcal{QA}_J^{\overline{p}}$ can be (*i.e.* satisfies all restrictions for) the context $Context_I^k$ of rule I, where $\overline{p} = 3 - p$.

The context of J is the multiset $Context_J = \mathcal{P}_I, \mathcal{E}_I \mathcal{QA}^2_I, Context^2_I, q\mathcal{A}'_I, \mathcal{C}''_I$. Condition ii) $\mathcal{P}_I = \emptyset$, $\mathcal{E}_I, \mathcal{QA}^{\overline{k}}_I, Context^{\overline{k}}_I = \mathcal{A}^1_I$ yields the following: $Context_J = \mathcal{P}_I, \mathcal{E}_I \mathcal{QA}^2_I, Context^2_I, q\mathcal{A}'_I, \mathcal{C}''_I = \mathcal{A}^1_I, q\mathcal{A}'_I, \mathcal{C}''_I$.
This fact and condition i) ensure the applicability of J on the appropriate (left) premise of I (*i.e.* fulfillment of condition $\boldsymbol{B_1}$ of Definition 3):

$$\dfrac{\underbrace{\vdash \mathcal{A}^1_I,\ \underbrace{\mathcal{QA}^1_I}_{\mathcal{A}'_J,\mathcal{QA}'_J,q\mathcal{A}'_I},\ \underbrace{Context^1_I}_{\mathcal{A}''_J,\mathcal{QA}''_J,\mathcal{C}''_I}}_{\vdash \underbrace{\mathcal{A}'_J,\mathcal{A}''_J}_{\mathcal{A}^1_J},\ \underbrace{\mathcal{QA}'_J,\mathcal{QA}''_J}_{\mathcal{QA}^1_J},\ \underbrace{\mathcal{A}^1_I, q\mathcal{A}'_I, \mathcal{C}''_I}_{Context_J}} \qquad \vdash \mathcal{A}^2_J,\ \mathcal{QA}^2_j,\ Context_J}{\vdash \mathcal{P}_J,\ \mathcal{QA}^1_J, \mathcal{QA}^2_J,\ Context_J,\ \underbrace{\mathcal{E}_J}_{\mathcal{E}'_J,\mathcal{A}'_J}}\ J$$

Further, condition iii) ensures that the multiset

$$\begin{aligned} \mathcal{P}_J, \mathcal{QA}^2_J, [Context^1_I, \mathcal{E}_j] \backslash \mathcal{A}^1_j &= \mathcal{P}_J, \mathcal{QA}^2_J, [\mathcal{E}'_J, \mathcal{C}''_I, \mathcal{QA}''_J, \mathcal{A}'_j, \mathcal{A}''_j] \backslash \mathcal{A}^1_j = \\ &= \mathcal{P}_J, \mathcal{QA}^2_J, \mathcal{E}'_J, \mathcal{C}''_I, \mathcal{QA}''_J \end{aligned}$$

satisfies all restrictions for the context in the corresponding (left *i.e.* $k = 1$) premise of rule I, which implies that the sequent

$$\underbrace{\vdash \mathcal{P}_J,\ \underbrace{\mathcal{QA}^1_J}_{\mathcal{QA}'_J,\mathcal{QA}''_J},\ \mathcal{QA}^2_J,\ \underbrace{Context_J}_{\mathcal{A}^1_I,q\mathcal{A}'_I,\mathcal{C}''_I},\ \underbrace{\mathcal{E}_J}_{\mathcal{E}'_J,\mathcal{A}'_J}}_{\vdash \mathcal{A}^1_I,\ \underbrace{\mathcal{QA}'_J, \mathcal{A}'_J, q\mathcal{A}'_I}_{\mathcal{QA}^1_I},\ \mathcal{P}_J, \mathcal{E}'_J, \mathcal{C}''_I, \mathcal{QA}''_J, \mathcal{QA}^2_J}$$

can be the corresponding (left) premise of I (this is condition $\boldsymbol{B_2}$ of Definition 3). Finally, we get the following derivation:

$$\dfrac{\underbrace{\dfrac{\underbrace{\vdash \mathcal{A}^1_I,\ \underbrace{\mathcal{QA}^1_I}_{\mathcal{A}'_J,\mathcal{QA}'_J,q\mathcal{A}'_I},\ \underbrace{Context^1_I}_{\mathcal{A}''_J,\mathcal{QA}''_J,\mathcal{C}''_I}}_{\vdash \underbrace{\mathcal{A}'_J,\mathcal{A}''_J}_{\mathcal{A}^1_J},\ \underbrace{\mathcal{QA}'_J,\mathcal{QA}''_J}_{\mathcal{QA}^1_J},\ \underbrace{\mathcal{A}^1_I, q\mathcal{A}'_I, \mathcal{C}''_I}_{Context_J}} \qquad \vdash \mathcal{A}^2_J, \mathcal{QA}^2_j, Context_J}{\vdash \mathcal{P}_J,\ \mathcal{QA}^1_J, \mathcal{QA}^2_J,\ Context_J,\ \underbrace{\mathcal{E}_J}_{\mathcal{E}'_J,\mathcal{A}'_J}}\ J}_{\vdash \mathcal{A}^1_I,\ \underbrace{\mathcal{QA}'_J, \mathcal{A}'_J, q\mathcal{A}'_I}_{\mathcal{QA}^1_I},\ \mathcal{P}_J, \mathcal{E}'_J, \mathcal{C}''_I, \mathcal{QA}''_J, \mathcal{QA}^2_J} \qquad \vdash \mathcal{A}^2_I, \mathcal{QA}^2_I, Context^2_I}{\vdash \mathcal{P}_I,\quad \mathcal{QA}^1_I, \mathcal{QA}^2_I,\quad \mathcal{P}_J, \mathcal{E}'_J, \mathcal{C}''_I, \mathcal{QA}''_J, \mathcal{QA}^2_J, Context^2_I,\quad \mathcal{E}_I}\ I$$

The equality of the end-sequents (*i.e.* condition C of Definition 3) is trivially true (and can be checked directly). Thus, the above derivation is the permutation result $\mathcal{PR}_{4.1}$ for I and J.
2) Similar to the proof of 1). $\square$

Theorem 2. *A strong additive rule I is permutable over a strong additive rule J yielding a permutation result of the form:*

1) $\mathcal{PR}_{4.k},\ k \in \{1,2\}$ 2) $\mathcal{PR}_{4.3}$

iff they satisfy the tests :

1) i) I and J are in strong permutation position;
ii) $\mathcal{P}_I = \emptyset,\ \ \mathcal{E}_I,\ \mathcal{QA}_I^{\overline{k}} = \mathcal{A}_I^k$;
iii) $\mathcal{P}_J = \emptyset,\ \ \mathcal{E}_J,\ \mathcal{QA}_J^2 = \mathcal{A}_J^1$;

2) i) I and J are in the strong-left and strong-right permutation positions;
ii) $\exists k \in \{1,2\}\ \ \mathcal{P}_I = \emptyset,\ \ \mathcal{E}_I,\ \mathcal{QA}_I^{\overline{k}} = \mathcal{A}_I^k$;
iii) The multiset $Context_J \backslash [\mathcal{P}_I, \mathcal{E}_I, \mathcal{QA}_I^k],\ \mathcal{A}_I^{\overline{k}}$ can be (i.e. satisfies all restrictions for) the context $Context_J$ of rule J.
iv) The multiset $[Context_I, \mathcal{E}_J] \backslash \mathcal{A}_J^1,\ \ \mathcal{P}_J,\ \mathcal{QA}_J^2$ can be (i.e. satisfies all restrictions for) the context $Context_I$ of rule I;
v) $[Context_I, \mathcal{E}_J] \backslash \mathcal{A}_J^1,\ \ \mathcal{P}_J,\ \mathcal{QA}_J^2 = [Context_I, \mathcal{E}_J] \backslash \mathcal{A}_J^2,\ \ \mathcal{P}_J,\ \mathcal{QA}_J^1$

Proof: Similar to the proof of Theorem 1. $\square$

Note that the occurrence of a rule in a derivation which satisfies either of the conditions $1)ii)$, $2)ii)$ of Theorem 1 or any of the conditions $1)ii)$, $1)iii)$, $2)ii)$ of Theorem 2, is actually a redundant step in that derivation. Informally, an instance of a rule can be considered as redundant in a given derivation if the conclusion of the rule instance is identical to one premise.

As an illustration, consider the following (linear logic) permutation of the multiplicative rule Cut and the strong additive rule $\&$. Note that in the example below the Cut rule is redundant (both before and after the permutation). Hence it is not necessary to permute or to try to permute these rules; the Cut instance can simply be eliminated.

$$\dfrac{\dfrac{\vdash A, \Gamma, C \quad \vdash A, A^{\perp}}{\vdash A, \Gamma, C}\,Cut \quad \vdash D, A, \Gamma}{\vdash C\&D, A, \Gamma}\,\& \quad \longleftrightarrow \quad \dfrac{\dfrac{\vdash A, \Gamma, C \quad \vdash D, A, \Gamma}{\vdash C\&D, A, \Gamma}\,\& \quad \vdash A, A^{\perp}}{\vdash C\&D, A, \Gamma}\,Cut$$

The proposed refined structure of sequent rules enables a precise specification of a redundant inference rule in a given derivation, as shown in the following definition.

Definition 4.
Any occurrence of a unary rule I in a derivation $\mathcal{D}$ is deemed to be a redundant step in $\mathcal{D}$ *iff* $\mathcal{A}_I = \mathcal{P}_I, \mathcal{E}_I$.
Any occurrence of a multiplicative (or strong additive) binary rule I in a derivation $\mathcal{D}$ is deemed to be redundant step in $\mathcal{D}$ *iff*
$\exists k \in \{1,2\}\ \mathcal{A}_I^k = \mathcal{P}_I, \mathcal{E}_I, \mathcal{QA}_I^{\overline{k}}, Context_I^{\overline{k}}$ *(or* $\exists k \in \{1,2\} \mathcal{A}_I^k = \mathcal{P}_I, \mathcal{E}_I, \mathcal{QA}_I^{\overline{k}}$ *)*
where $\overline{k} = 3 - k$.

In [16] (Theorem 4.6), the authors of this paper proved that an instance of a unary rule which can be permuted over a strong additive rule must be a redundant step in a derivation. Thus, if the strategy of *"eliminate redundant steps in a given derivation prior to any other proof transformation i.e. prior to any permutation"*, is adopted, then as a consequence of Theorem 1, Theorem 2 and Theorem 4.6 (from [16]) the following assertion holds:

It is never possible to permute one of the rules over a strong additive rule.[13]

The form of the result of a permutation can be determined in advance, independently of the concrete analysis of rule instances in a proof, as specified in Theorem 3 below. The result is appropriate for a process such as adapting proofs to obey certain strategies, identifying cases where this is not possible, minimizing the amount of branching in the proof, delaying certain choices until the availability of an optimum amount of information, and controlling the degree of non-determinism (if possible, so as not to duplicate rules or branches with a high level of non-determinism).

Theorem 3.
If redundant steps in a given derivation are eliminated prior to any other proof transformation (i.e. prior to any permutation) then, for a pair of adjacent rules I and J the following assertions hold:

1) Any permutation of a strong additive rule I over a unary rule J results only in the permutation result of the form shown on the right-hand side below:

$$\mathcal{ID}\quad \dfrac{\dfrac{\begin{matrix}\vdots\,\mathcal{D}_1 & \vdots\,\mathcal{D}_2\\ \gamma_1 & \gamma_2\end{matrix}}{\beta}\,I}{\alpha}\,J \qquad \longmapsto \qquad \mathcal{PR}_{3.3}\quad \dfrac{\dfrac{\begin{matrix}\vdots\,\mathcal{D}_1\\ \gamma_1\end{matrix}}{\beta_1}\,J \quad \dfrac{\begin{matrix}\vdots\,\mathcal{D}_2\\ \gamma_2\end{matrix}}{\beta_2}\,J}{\alpha}\,I$$

2) Any permutation of a strong additive binary rule over a unary rule always implies duplication of the corresponding unary rule.

3) Any permutation of a strong additive rule I over a multiplicative binary rule J results only in the permutation result of the form as shown on the right-hand side below:

$$\mathcal{ID}\quad \dfrac{\dfrac{\begin{matrix}\vdots\,\mathcal{D}_1 & \vdots\,\mathcal{D}_2\\ \gamma_1 & \gamma_2\end{matrix}}{\beta_1}\,I \quad \begin{matrix}\vdots\,\mathcal{D}_3\\ \beta_2\end{matrix}}{\alpha}\,J \qquad \longmapsto \qquad \mathcal{PR}_{4.3}\quad \dfrac{\dfrac{\begin{matrix}\vdots\,\mathcal{D}_1 & \vdots\,\mathcal{D}_3\\ \gamma_1 & \beta_2\end{matrix}}{\delta_1}\,J \quad \dfrac{\begin{matrix}\vdots\,\mathcal{D}_2 & \vdots\,\mathcal{D}_3\\ \gamma_2 & \beta_2\end{matrix}}{\delta_2}\,J}{\alpha}\,I$$

4) Any permutation of a strong additive binary rule over a multiplicative binary rule always implies duplication of certain proof branches.

5) Any permutation of a multiplicative rule I over a unary rule J may produce the permutation result of the form shown on the right-hand side below:

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[13] A standard permutation is assumed (*i.e.* swapping the order of $\underline{two}$ adjacent inferences) in a derivation $\mathcal{D}$ while preserving the root sequent and the set of leaves of $\mathcal{D}$.

$$\mathcal{ID}\quad \dfrac{\dfrac{\begin{array}{c}\vdots\,\mathcal{D}_1\\ \gamma_1\end{array}\quad\begin{array}{c}\vdots\,\mathcal{D}_2\\ \gamma_2\end{array}}{\beta}\,I}{\alpha}\,J \quad\longmapsto\quad \begin{cases} \mathcal{PR}_{3.1}\ \dfrac{\dfrac{\begin{array}{c}\vdots\,\mathcal{D}_1\\ \gamma_1\end{array}}{\beta_1}\,J \quad \begin{array}{c}\vdots\,\mathcal{D}_2\\ \gamma_2\end{array}}{\alpha}\,I \quad \text{or} \quad \mathcal{PR}_{3.2}\ \dfrac{\begin{array}{c}\vdots\,\mathcal{D}_1\\ \gamma_1\end{array}\quad \dfrac{\begin{array}{c}\vdots\,\mathcal{D}_2\\ \gamma_2\end{array}}{\beta_2}\,J}{\alpha}\,I\end{cases}$$

6) Any permutation of a multiplicative rule I over a multiplicative binary rule J may produce the permutation result of the form shown on the right-hand side below:

$$\mathcal{ID}\quad \dfrac{\dfrac{\begin{array}{c}\vdots\,\mathcal{D}_1\\ \gamma_1\end{array}\quad\begin{array}{c}\vdots\,\mathcal{D}_2\\ \gamma_2\end{array}}{\beta_1}\,I \quad \begin{array}{c}\vdots\,\mathcal{D}_3\\ \beta_2\end{array}}{\alpha}\,J \quad\longmapsto\quad \begin{cases} \mathcal{PR}_{4.1}\ \dfrac{\dfrac{\begin{array}{c}\vdots\,\mathcal{D}_1\\ \gamma_1\end{array}\quad\begin{array}{c}\vdots\,\mathcal{D}_3\\ \beta_2\end{array}}{\delta_1}\,J \quad \begin{array}{c}\vdots\,\mathcal{D}_2\\ \gamma_2\end{array}}{\alpha}\,I \quad \text{or} \quad \mathcal{PR}_{4.2}\ \dfrac{\begin{array}{c}\vdots\,\mathcal{D}_1\\ \gamma_1\end{array}\quad \dfrac{\begin{array}{c}\vdots\,\mathcal{D}_2\\ \gamma_2\end{array}\quad\begin{array}{c}\vdots\,\mathcal{D}_3\\ \beta_2\end{array}}{\delta_2}\,J}{\alpha}\,I\end{cases}$$

3.2. Special Permutations

As noted in Subsection 4.2 of [10], there are certain non-permutabilities which can be overcome by taking special cases into account. For a pair of adjacent, non-permutable rules I and J (where J is a binary rule), one special case is when the rule I is not just above the left premise of the lower rule J, but above both premises. For example, for a unary rule I the idea is to note the derivation shown on the left (if any) and to (try to) transform it into the right-hand derivation as follows:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\mathcal{D}_1\\ \alpha\end{array}}{\gamma_1}\,I \quad \dfrac{\begin{array}{c}\vdots\,\mathcal{D}_2\\ \beta\end{array}}{\gamma_2}\,I}{\delta}\,J \quad\longmapsto\quad \dfrac{\dfrac{\begin{array}{c}\vdots\,\mathcal{D}_1\\ \alpha\end{array}\quad\begin{array}{c}\vdots\,\mathcal{D}_2\\ \beta\end{array}}{\gamma}\,J}{\delta}\,I$$

Such transformations are referred to as *special permutations*.

Definition 5.

Transformations of the following form

$i)$
$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\mathcal{D}_1\\ \gamma_1\end{array}}{\beta_1}\,I \quad \dfrac{\begin{array}{c}\vdots\,\mathcal{D}_2\\ \gamma_2\end{array}}{\beta_2}\,I}{\alpha}\,J \quad\longmapsto\quad \dfrac{\dfrac{\begin{array}{c}\vdots\,\mathcal{D}_1\\ \gamma_1\end{array}\quad\begin{array}{c}\vdots\,\mathcal{D}_2\\ \gamma_2\end{array}}{\beta}\,J}{\alpha}\,I$$

$ii)$
$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\mathcal{D}_1\\ \gamma_1\end{array}\quad\begin{array}{c}\vdots\,\mathcal{D}_2\\ \gamma_2\end{array}}{\delta_1}\,I \quad \dfrac{\begin{array}{c}\vdots\,\mathcal{D}_3\\ \beta_1\end{array}\quad\begin{array}{c}\vdots\,\mathcal{D}_4\\ \beta_2\end{array}}{\delta_2}\,I}{\alpha}\,J \quad\longmapsto\quad \dfrac{\dfrac{\begin{array}{c}\vdots\,\mathcal{D}_1\\ \gamma_1\end{array}\quad\begin{array}{c}\vdots\,\mathcal{D}_3\\ \beta_1\end{array}}{\delta_1'}\,J \quad \dfrac{\begin{array}{c}\vdots\,\mathcal{D}_2\\ \gamma_2\end{array}\quad\begin{array}{c}\vdots\,\mathcal{D}_4\\ \beta_2\end{array}}{\delta_2'}\,J}{\alpha}\,I$$

$iii)$
$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\mathcal{D}_1\\ \gamma_1\end{array}\quad\begin{array}{c}\vdots\,\mathcal{D}\\ \beta_2\end{array}}{\delta_1}\,I \quad \dfrac{\begin{array}{c}\vdots\,\mathcal{D}_2\\ \beta_1\end{array}\quad\begin{array}{c}\vdots\,\mathcal{D}\\ \beta_2\end{array}}{\delta_2}\,I}{\alpha}\,J \quad\longmapsto\quad \dfrac{\dfrac{\begin{array}{c}\vdots\,\mathcal{D}_1\\ \gamma_1\end{array}\quad\begin{array}{c}\vdots\,\mathcal{D}_2\\ \beta_1\end{array}}{\beta_1'}\,J \quad \begin{array}{c}\vdots\,\mathcal{D}\\ \beta_2\end{array}}{\alpha}\,I$$

are referred to as special permutations of I over J.[14].

The objective is to resolve the non-permutabilities associated with an upward permutation of strong-additive rules. In [16], the authors specified all the necessary and sufficient conditions (Theorem 4.6) for special permutation of a unary rule over a strong additive rule. It is not difficult to apply the same approach and analyse special permutations of an arbitrary rule I over a strong additive rule J. For example, Theorem 4 provides a necessary condition for special permutation of a multiplicative rule over a strong additive rule.

Theorem 4.

If the special permutation of a multiplicative rule I *over a strong additive rule* J *of the following form*

$$\dfrac{\dfrac{\overset{\vdots\,\mathcal{D}_1}{\alpha_1} \quad \overset{\vdots\,\mathcal{D}_2}{\alpha_2}}{\gamma_1}\,I \qquad \dfrac{\overset{\vdots\,\mathcal{D}_3}{\beta_1} \quad \overset{\vdots\,\mathcal{D}_4}{\beta_2}}{\gamma_2}\,I}{\delta}\,J \quad\longmapsto\quad \dfrac{\dfrac{\overset{\vdots\,\mathcal{D}_1}{\alpha_1} \quad \overset{\vdots\,\mathcal{D}_3}{\beta_1}}{\gamma_1'}\,J \qquad \dfrac{\overset{\vdots\,\mathcal{D}_2}{\alpha_2} \quad \overset{\vdots\,\mathcal{D}_4}{\beta_2}}{\gamma_2'}\,J}{\delta}\,I$$

is possible, then the occurrence of J *in the starting (i.e. left-hand) derivation is a redundant step.*

Proof: Similar to the proof of Theorem 1. □

As a consequence of Theorem 4, if the redundant steps in a given derivation are eliminated prior to any other proof transformation (*i.e.* prior to any permutation/special permutation), it is not possible to rearrange the derivation as shown in the above theorem. However, special permutation of the following form is possible:

$$\dfrac{\dfrac{\overset{\vdots\,\mathcal{D}_1}{\alpha_1} \quad \overset{\vdots\,\mathcal{D}}{\alpha}}{\gamma_1}\,I \qquad \dfrac{\overset{\vdots\,\mathcal{D}_3}{\beta_1} \quad \overset{\vdots\,\mathcal{D}}{\alpha}}{\gamma_2}\,I}{\delta}\,J \quad\longmapsto\quad \dfrac{\dfrac{\overset{\vdots\,\mathcal{D}_1}{\alpha_1} \quad \overset{\vdots\,\mathcal{D}_3}{\beta_1}}{\gamma}\,J \qquad \overset{\vdots\,\mathcal{D}}{\alpha}}{\delta}\,I,$$

as illustrated by the linear logic example below. This special permutation can be specified in a way similar to that shown in Theorem 1 and Theorem 2.

$$\dfrac{\dfrac{\overset{\vdots\,\mathcal{D}_1}{\vdash A, C, \Delta} \quad \overset{\vdots\,\mathcal{D}}{\vdash B, \Gamma}}{\vdash A \otimes B, C, \Delta, \Gamma}\,\otimes \qquad \dfrac{\overset{\vdots\,\mathcal{D}_3}{\vdash D, A, \Delta} \quad \overset{\vdots\,\mathcal{D}}{\vdash B, \Gamma}}{\vdash A \otimes B, D, \Delta, \Gamma}\,\otimes}{\vdash C \& D, A \otimes B, \Delta, \Gamma}\,\& \quad\longleftrightarrow\quad \dfrac{\dfrac{\overset{\vdots\,\mathcal{D}_1}{\vdash A, C, \Delta} \quad \overset{\vdots\,\mathcal{D}_3}{\vdash D, A, \Delta}}{\vdash C \& D, A, \Delta}\,\& \qquad \overset{\vdots\,\mathcal{D}}{\vdash B, \Gamma}}{\vdash C \& D, A \otimes B, \Delta, \Gamma}\,\otimes$$

[14] Note that a special permutation assumes that the resulting derivation has the same end-sequent and the same set of leaves as the starting derivation.

4. Conclusions and Future Work

Permutation of inference rules can be treated in a general way by distinguishing the formula roles in inference rules. This paper showed how the finer distinction of formula roles, introduced in [16], can be used to specify (and prove) certain dependencies between the ways of permuting rules and their generic properties. For example, the *upward permutation* of a strong-additive rule requires a set of particular conditions. Also, the possible forms of proof after permutation depend on the generic properties of the rules in question and, therefore, can be determined independently of direct permutation. One outcome of such analysis is an automated-oriented specification of necessary and sufficient conditions for the permutation of given rules.

The results outlined here are directly applicable to a number of sequent calculus formalizations of a range of logic such as, sequent calculus for classical logic (LK [8]), single-conclusioned sequent calculus for intuitionistic logic (LJ [8], LJT [6]), multiple-conclusioned sequent calculus for intuitionistic logic (LJT multiple conclusioned [6], LM [29]) , sequent calculus for classical linear logic ([11], [14], [27], [24]), sequent calculus for intuitionistic linear logic ($\mathcal{ILL}$ [28], ILL [12], $\mathcal{L}$ [12]), sequent calculus for modal logic (S4 [29]), and sequent calculus for Lax Logic (PLL [7]).

The authors believe that the proposed approach in permutation analysis can be *extended and adapted* to address permutations of sequent rules that assume: commutative and associative contexts and rules with no more than two premises, as well as rules with multi-zones[1].

However, the permutability still needs further, finer categorization of the sequent structure, for a complete analysis. As an example, consider the following two rules from Intuitionistic Light Affine Logic ([2]) and the permutation shown on the right hand side below:

$$\frac{\Gamma \vdash \Delta}{\Gamma, \phi \vdash \Delta}\ \textit{Weak.} \qquad \frac{\Delta, \Gamma \vdash C}{!\Delta, \S\Gamma \vdash \S C}\ \S \qquad \cfrac{\cfrac{\begin{array}{c}\vdots\\ P, \Gamma \vdash C\end{array}}{P, A, \Gamma \vdash C}\ \textit{Weak.}}{!P, !A, \S\Gamma \vdash \S C}\ \S \quad \rightsquigarrow \quad \cfrac{\cfrac{\begin{array}{c}\vdots\\ P, \Gamma \vdash C\end{array}}{!P, \S\Gamma \vdash \S C}\ \S}{!P, !A, \S\Gamma \vdash \S C}\ \textit{Weak.}$$

The rules *Weak* and $\S$ are permutable even though they are not in a strong permutation position in the starting subproof.

The $extra$ formula ($Extra_{Weak.} = \{ A \}$) of the $Weak.$ rules a crucial 'role' in the above permutation, as does the fact that only the active formula C is an obligatory active formula for the $\S$ rule ($Active_{\S} = \{ A, P, \Gamma, C \}$). This indicates that further analysis of $active$ formulae (*i.e.* distinction between of the obligatory and non-obligatory formulae) is needed.

A further natural extension of this work is to attempt to apply the proposed refined categorization of sequent rules to exploit some similarities in the structure of sequent calculus inference rules from different proof systems for the extension and generalization of certain methods and techniques originally designed for particular sets of rules.

The proposed refined structure of sequent rules enables a formal specification and generalization of the proposed constraints for the identification of redundant formulae. Informally, a (sub)formula can be considered as redundant in a given proof if the deletion of the (sub)formula from the proof still results in a provable sequent. In [17], the authors have identified several classes of rules which require special attention from the perspective of redundant formulae. An analysis of the representatives of each class of rules and the corresponding constraints, in terms of the proposed refined structure of inference rules, is an item of further research.

The authors offer this work as a contribution to the library [19] of automatic support tools for reasoning about (permutations in) sequent calculi proof search, trusting that the results of this analysis can then be implemented and utilized by means of an automated proof assistant such as, Twelf [25, 26], possibly in conjunction with constraint logic programming techniques [18], [29].

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Tatjana Lutovac is an Assistant Professor at the Department of Applied Mathematics, School of Electrical Engineering, University of Belgrade. She received her PhD degree at School of Computer Science and Information Technology, RMIT University in Melbourne, Australia. Her research interests include computational logic, automated deduction, verification, logical frameworks, design and implementation of programming languages, logic programming, data mining.

James Harland is an Associate Professor in Computational Logic in the School of Computer Science and Information Technology at RMIT University in Melbourne, Australia. He has a PhD from the University of Edinburgh, and an Honours degree in Mathematics and Computer Science from the University of Melbourne. His research interests include resource-sensitive logics, proof theory, automated reasoning, intelligent agents, agent programming languages, intelligent narratives, computer science education and Turing machines.

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