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Convexity of hesitant fuzzy sets based on aggregation functions

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Abstract. Convexity is one of the most important geometric properties of sets and a useful concept in many fields of mathematics, like optimization. As there are also important applications making use of fuzzy optimization, it is obvious that the studies of convexity are also frequent. In this paper we have extended the notion of convexity for hesitant fuzzy sets in order to fulfill some necessary properties. Namely, we have found an appropriate definition of convexity for hesitant fuzzy sets on any ordered universe based on aggregation functions such that it is compatible with the intersection, that is, the intersection of two convex hesitant fuzzy sets is a convex hesitant fuzzy set and it fulfills the cutworthy property.

Keywords: hesitant fuzzy set, alpha-cut, aggregation function, convexity.

1. Introduction

Convexity is a basic mathematical notion that has been used to analyse many different problems. Its practical applications in several areas are very important, like optimization [20], image processing [32], robotics [19] or geometry [15], among many others.

Since most of practical problems include approximate information, fuzzy convexity has been studied in deep in the literature. Thus, several types of convexity of fuzzy sets were studied by different authors (see, for instance, Ammar and Metz [2], Diaz et al. [10], Ramik and Vlach [25], Sarkar [29], Syau and Lee [31] and Yang [38]).

The necessity of dealing with imprecision in real world problems has been a long-term research challenge that has originated different extensions of fuzzy sets. A special case of type-2 fuzzy sets are the hesitant fuzzy sets. They can be considered as an extension of fuzzy sets different from Atanassov's intuitionistic fuzzy sets ([3]) or interval-valued fuzzy sets (introduced independently by Zadeh [40], Grattan-Guiness [11], Jahn [12], Sambuc [28] in the seventies). Hesitant fuzzy sets can be useful to deal with situations where the previous tools are not so efficient as, for instance, the modeling of an evaluation by a group of experts, when it is not possible or easy to obtain a consensus to unify the different opinions. Hesitant fuzzy sets were formally introduced by Torra [33], but the idea behind this concept was already considered previously, as we can see in Grattan-Guinness [11]. Although their definition is relatively new, it has attracted very quickly the attention of many researchers, since they could see the high potential of them for applications, especially in decision making, as we can see in [34,35,37,41]. Perhaps by this reason, several concepts, tools and trends related to this extension have to be studied. Taking into account the previous comments, we are specially interested in the concept of convex hesitant fuzzy sets. As far as we know, the first attempt to define convexity for hesitant fuzzy sets has been done by Rashid and Beg in 2016 [26]. That definition had some problems, which were solved by Janiš et al. in 2018 [14]. These two definitions seem to be quite different. However, we can notice that they are really related, since both of them are based on aggregation functions. Thus, Rashid and Beg considered the arithmetic mean and Janiš et al. considered the classical t-conorm of the maximum. Then, both definitions can be considered as particular cases of a general convexity based on aggregation functions. The main aim of this paper is to introduce a general definition and study its properties. In particular, we are going to study in depth the preservation of convexity for alpha-cuts (the cutworthy property) and under intersections, since the first one is a very important property in fuzzy set theory and the second one is a necessary property in many applications, as optimization. Thus, we will try to characterize the behavior of aggregation functions with respect to both properties.

The remainder of this paper is organized as follows. In Section 2, some basic concepts about convexity for fuzzy sets and hesitant fuzzy sets are recalled and the notation is fixed. Section 3 is devoted to the new definition of convexity for hesitant fuzzy sets based on an aggregation function with a detailed study of the preservation of convexity under intersections. In Section 4 we present an example from the area of optimization. Finally, some conclusions and open problems are formulated in Section 5.

2. Basic concepts

It is well-known that for any nonempty set X, usually called the universe, Zadeh defined a fuzzy set A in X by means of the map $\mu_A : X \to [0,1]$, which is said to be the membership function of A (see [39]). Thus, a way to describe the fuzzy set could be $A = \{\langle x, \mu_A(x)\rangle : x \in X\}$.

For any $\alpha \in (0,1]$, a crisp subset of X is associated to A as follows: $A_\alpha = \{x \in X : \mu_A(x) \geq \alpha\}$. This set is called the α-cut of A and the collection of all the alpha-cuts totally characterizes the fuzzy set.

A particular case of fuzzy sets are the convex fuzzy sets. A crisp subset A of a linear space X is convex if and only if $\lambda x + (1-\lambda)y \in A$ for any $x, y \in A$ and for any $\lambda \in [0,1]$ (for a detalied study on convex set see e.g. [18]). From this definition, a natural extension for fuzzy sets could be to require that the fuzzy set A fulfills the property: $\mu_A(\lambda x + (1-\lambda)y) \geq \lambda\mu_A(x) + (1-\lambda)\mu_A(y)$, for all $x, y \in X$ and for all $\lambda \in [0,1]$. However, this definition was not considered from the beginning, since as already Zadeh noticed in [39], there is not an equivalence between the convexity of the alpha-cuts and the convexity of the fuzzy set.

In order to solve this problem, the first definition of convex fuzzy set considered in that paper was that

$$\mu_A(\lambda x + (1-\lambda)y) \geq \min\{\mu_A(x), \mu_A(y)\}$$

for any $x, y \in X$ and any $\lambda \in [0,1]$ (see [39]). It is known that A is a convex fuzzy set if and only if A_α is a convex crisp set, for any $\alpha \in (0,1]$, that is, the cutworthy property is fulfilled. Apart from that, this concept is preserved by the intersection, that is, if A and B are two convex fuzzy sets, then $A \cap B$ is a convex fuzzy set, with the classical definition of intersection introduced by Zadeh. Another advantage of this definition is that the addition that appears on the right side of the inequality could make no sense when working in a more general environment (e.g. lattice-valued fuzzy sets), where such operation is not defined in general.

As we commented at the introduction, several generalizations of fuzzy sets have been considered in the literature. In this paper we are interested in the class of hesitant fuzzy sets. Now, instead of a single number, the membership function returns a set of membership values for each element in the domain. More precisely:

Definition 1 *[33,36] Let X be a universe. A hesitant fuzzy set on X is defined by means of a function $h_A : X \to \mathcal{P}([0,1])$, where $\mathcal{P}([0,1])$ denotes the power set of the interval $[0,1]$, such that for any element of X it returns a subset of the unit interval $[0,1]$. This can be represented by*

$$A = \{\langle x, h_A(x)\rangle : x \in X\},$$

where $h_A(x)$ is a set of values in $[0,1]$, denoting the possible membership degrees of the element $x \in X$ to the set A.

In this definition, any subset of the interval $[0,1]$ could be the membership degree for an element in X. However, some particular cases of specific subsets are the most important in the literature. Thus, for instance, the case of interval-valued hesitant fuzzy sets has been studied lately (see, for instance, [7]). However, it was already noted in [4] that in practical situations we deal frequently with only finite subsets. Thus, in most of the cases, the assumption that the membership degrees are finite and nonempty subsets is considered. Then we are able to work with a particular type of hesitant fuzzy sets, the typical hesitant fuzzy sets, defined as follows:

Definition 2 *[4] Let $\mathcal{H} \subset \mathcal{P}([0,1])$ be the set of all finite nonempty subsets of the interval $[0,1]$ and let X be a nonempty universe. A typical hesitant fuzzy set A over X is given by*

$$A = \{\langle x, h_A(x)\rangle : x \in X\},$$

where $h_A : X \to \mathcal{H}$.

Throughout this work we will deal only with typical hesitant fuzzy sets and, by simplicity, we will call them just hesitant fuzzy sets.

Several authors have studied different operations on hesitant fuzzy sets (see, e.g., [24,33,36]). In particular, a very important concept in this work is the intersection of two hesitant fuzzy sets proposed by Torra ([33]), that extends the classical definition of intersection of two fuzzy sets given by Zadeh ([39]).

Definition 3 *[33] Let A, B be hesitant fuzzy sets on a universe X. The intersection of A and B is a hesitant fuzzy set, denoted by $A \cap B$ defined by:*

$$h_{A\cap B}(x) = \{\gamma \in \{h_A(x) \cup h_B(x)\} \,|\, \gamma \leq \min\{\max\{h_A(x)\}, \max\{h_B(x)\}\}\}$$

for any $x \in X$.

Thus, the membership function of the intersection is obtained as the set of all values in any of the two sets which are lower than or equal to the smaller of the two maximums at a particular point. In order to clarify this definition, we will show an easy example.

Example 1 *If we consider an ordered space $X = \{x_1, x_2, x_3\}$ with $x_1 < x_2 < x_3$ and the hesitant fuzzy sets A and B defined by*

X	x_1	x_2	x_3
A	$\{0.2, 0.6\}$	$\{0\}$	$\{0.2, 0.4, 0.6, 0.8\}$
B	$\{0.4\}$	$\{0.6\}$	$\{1\}$

their intersection is defined as

X	x_1	x_2	x_3
$A \cap B$	$\{0.2, 0.4\}$	$\{0\}$	$\{0.2, 0.4, 0.6, 0.8\}$

The graphical representation of A, B and $A \cap B$ is in Figure 1.

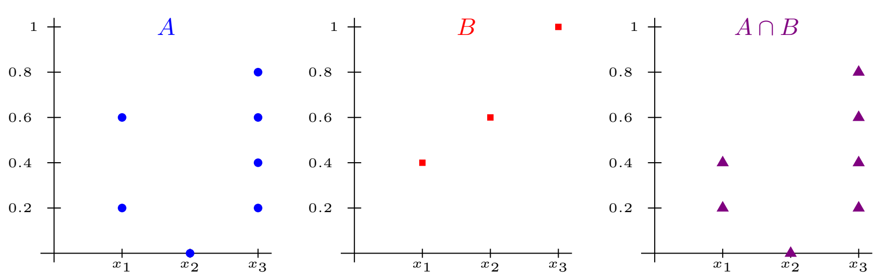


Fig. 1. Example of intersection of hesitant fuzzy sets.

The hesitant fuzzy logic associated to this operation and the corresponding union can be encompassed in the logic systems proposed in [8] by considering the appropriate lattices and set of functions. As a consequence of the concepts introduced in [9], we could consider some different approach to the idea of intersection of two hesitant fuzzy sets, but we have prefered to consider the usual definition of intersection, in order to be able to link our research with the related recent papers in the literature [1,5,14,24,26,27].

Alcantud [1] and Zhu et al. [42] used this definition of intersection on their works in order to define new intersections between dual hesitant fuzzy elements and dual extended hesitant fuzzy elements, respectively. Similarly, Pei and Yi [24] also used this definition to investigate about semilattices of hesitant fuzzy sets. Rodriguez et al. [27] presented an overview of hesitant fuzzy sets to provide a clear perspective on the different concepts, tools, and trends related to this extension of fuzzy sets. On [26], Rashid and Beg provided a definition of convexity based on the convexity of the score function that does not guarantee the preservation of convexity under intersections, but Janis et al. [14] presented a concept of convexity for hesitant fuzzy sets, based on the maximum operator, without this drawback.

Until now we were considering hesitant fuzzy sets in general, but we are particularly interested in convex (typical) hesitant fuzzy sets. The first definition given at the literature was based on the score function, which was introduced by Xia and Xu [36].

Definition 4 *Let A be a hesitant fuzzy set on a finite universe X. The map $s_A : X \to [0,1]$ defined by*

$$s_A(x) = \frac{1}{|h_A(x)|} \sum_{\gamma \in h_A(x)} \gamma$$

is called the score function of A.

It is clear the the score function is just the arithmetic mean of the membership function h_A of A at any point x in X. Based on this concept, Rashid and Beg introduced in 2016 the concept of convexity for hesitant fuzzy sets.

Definition 5 *[26] Let X be a finite linear space. A hesitant fuzzy set A on the universe X is said to be convex, if for all $x, y \in X$, and $\lambda \in [0,1]$ it holds that*

$$s_A(\lambda x + (1-\lambda)y) \geq \min\{s_A(x), s_A(y)\}.$$

The authors in fact call this property quasiconvexity instead of convexity, but we will use the previous name for simplicity. They also considered that for any hesitant fuzzy set, its alpha-cut is defined by:

$$A_\alpha = \{x \in X : s_A(x) \geq \alpha\}$$

for any $\alpha \in (0,1]$ and with this definition they prove the following result.

Proposition 1 *[26] Let X be a finite linear space and let A be a hesitant fuzzy set defined on X. The followings statements are equivalent:*

1. *A is a quasi-convex hesitant fuzzy set.*
2. *Any α-cut of A is convex crisp set.*

Apart from the cutworthy approach, one of the principal properties of convexity for fuzzy sets was its preservation under arbitrary intersections. This is a very important requirement, since it makes the collection of convex sets very important for applications as, for instance, optimization (see [2] or [30]). As it was shown in [14], the concept of convexity introduced by Rashid and Beg does not preserve this property. In that paper, a definition of convexity for hesitant fuzzy sets was given such that it fulfills the natural conditions: it extends the concept of convexity for fuzzy sets, it preserves convexity under intersections and equivalence of the convexity for cuts. Taking into account these properties, they arrived to a natural definition of convexity for hesitant fuzzy sets based on the maximum of the membership values at any point. More precisely,

Definition 6 *[14] Let X be a finite linear space and let A be a hesitant fuzzy set on X. Then A is convex, if*

$$\max\{h_A(\lambda x + (1-\lambda)y)\} \geq \min\{\max\{h_A(x)\}, \max\{h_A(z)\}\}$$

for each $x, z \in X$ and for each $\lambda \in [0,1]$.

Although the definitions 5 and 6 are not so similar at a first sight, they have a common idea behind. We define convexity based on the arithmetic mean or the maximum. Thus, in both cases, we use two particular examples of aggregation functions. Let us recall this concept.

Definition 7 *[21] Let $\mathcal{A} : \cup_{i=1}^{n}[0,1]^i \to [0,1]$ such that*

- $\mathcal{A}(0,0,\ldots,0) = 0, \mathcal{A}(1,1,\ldots,1) = 1$
- $\mathcal{A}(x) = x$ *for all* $x \in [0,1]$
- $\mathcal{A}$ *is monotone at each variable*

then $\mathcal{A}$ is called an aggregation function.

This will be the starting point for our general definition of convexity for hesitant fuzzy sets.

3. General convexity for hesitant fuzzy sets

Once we have defined an aggregation function, we are able to formulate convexity for any hesitant fuzzy set defined on a finite universe. In order to consider the most general possible definition, we will also consider any ordered space as the universe, instead of a linear space. This is not a real generalization, but it could be more appropriate at the environment we use to work.

Definition 8 *Let X be an ordered space, let A be a hesitant fuzzy set on X, let $\mathcal{A}$ be an aggregation function. Then A is $\mathcal{A}$-convex, if for each $x < y < z$ there is*

$$\mathcal{A}(h_A(y)) \geq \min\{\mathcal{A}(h_A(x)), \mathcal{A}(h_A(z))\}.$$

By the second axiom in Definition 7, it is immediate that a convex fuzzy set considered as a hesitant fuzzy set with singleton values is convex.

Another usual requirement is the cutworthy property. Thus, first of all, we will propose a reasonable definition of cut for hesitant fuzzy sets.

Definition 9 *Let X be an ordered space, let A be a hesitant fuzzy set on X, let $\mathcal{A}$ be an aggregation function. The α-cut of A with respect to $\mathcal{A}$ is defined as the crisp set:*

$$A_\alpha^{\mathcal{A}} = \{x \in X : \mathcal{A}(h_A(x)) \geq \alpha\}$$

for any $\alpha \in (0,1]$.

With respect to this definition, convexity of a hesitant fuzzy set is equivalent to convexity of its cuts.

Proposition 2 *Let X be an ordered space, let A be a hesitant fuzzy set on X, let $\mathcal{A}$ be an aggregation function. A is $\mathcal{A}$-convex if and only if $A_\alpha^{\mathcal{A}}$ is convex for any $\alpha \in (0,1]$.*

Proof. Let A be a hesitant fuzzy set and let x, y, z be elements in X with $x < y < z$.

On one hand, let us consider that A is $\mathcal{A}$-convex. If $x, z \in A_\alpha^{\mathcal{A}}$ and $\alpha \in (0, 1]$, we know that $\mathcal{A}(h_A(x)), \mathcal{A}(h_A(z)) \geq \alpha$. Thus, by the $\mathcal{A}$-convexity of A, we know that $\mathcal{A}(h_A(y)) \geq \min\{\mathcal{A}(h_A(x)), \mathcal{A}(h_A(z))\} \geq \alpha$, that is, $A_\alpha^{\mathcal{A}}$ is a convex crisp set.

On the other hand, if the α-cuts of A w.r.t. $\mathcal{A}$ are convex for any $\alpha \in (0, 1]$, we can consider in particular the value $\alpha_0 = \min\{\mathcal{A}(h_A(x)), \mathcal{A}(h_A(z))\}$. It is clear that $x, z \in A_{\alpha_0}^{\mathcal{A}}$. By the convexity of this set, we have that $y \in A_{\alpha_0}^{\mathcal{A}}$ and, therefore, $\mathcal{A}(h_A(y)) \geq \alpha_0$. By taking into account the definition of α_0, we have that A is an $\mathcal{A}$-convex hesitant fuzzy set. □

Thus, this definition generalizes the idea of convexity for fuzzy sets and it is equivalent to the convexity of its associated cuts. The remaining natural property is the preservation of convexity under intersections. Thus, we are going to study whether the intersection of convex hesitant fuzzy sets is a convex hesitant fuzzy set. As we will see, this property is not fulfilled in general. Therefore our main aim is to characterize those aggregations, that lead to a class of convex hesitant fuzzy sets, for which the convexity of their intersections is preserved. To shorten our formulations, we will use the notion "$\mathcal{A}$ preserves convexity", if the class of convex hesitant fuzzy sets obtained using $\mathcal{A}$ in Definition 8 (we have called them $\mathcal{A}$-convex) is closed under intersections.

A similar question of preserving convexity under aggregation for fuzzy sets has been solved in [13].

We will distinguish several cases for the aggregation function $\mathcal{A}$, namely the following ones:

1. $\mathcal{A}$ is the maximum or the minimum.
2. $\mathcal{A}$ is lower than $\min$ at some point.
3. $\mathcal{A}$ is greater than $\max$ at some point.
4. $\mathcal{A}$ is between $\min$ and $\max$ but it is different from both, maximum and minimum.

3.1. The case of the maximum/minimum

If we choose $\mathcal{A}(a, b) = \max\{a, b\}$ for all $a, b \in [0, 1]$ as the aggregation function, convexity is preserved and the same happens for $\mathcal{A}$ being the minimum.

Proposition 3 *Let X be an ordered space. If $\mathcal{A}(a, b) = \max\{a, b\}$ for all $a, b \in [0, 1]$, then $\mathcal{A}$ preserves convexity.*

Proof. Let A and B be two $\mathcal{A}$-convex hesitant fuzzy sets. Let $x, y, z \in X$ such that $x \leq y \leq z$. For any $t \in X$ we have

$$\begin{aligned}\mathcal{A}(h_{A\cap B}(t)) &= \max\{h_{A\cap B}(t)\} = \min\{\max\{h_A(t)\}, \max\{h_B(t)\}\} \\ &= \min\{\mathcal{A}(h_A(t)), \mathcal{A}(h_B(t))\}\end{aligned}$$

by taking into account the definition of $\mathcal{A}$ and Definition 3. Thus,

$$\mathcal{A}(h_{A\cap B}(y)) = \min\{\mathcal{A}(h_A(y)), \mathcal{A}(h_B(y))\}$$

and by the A-convexity of A and B,

$$\mathcal{A}(h_{A\cap B}(y)) \geq \min\{\min\{\mathcal{A}(h_A(x)), \mathcal{A}(h_A(z))\}, \min\{\mathcal{A}(h_B(x)), \mathcal{A}(h_B(z))\}\} =$$

$$\min\{\min\{\mathcal{A}(h_A(x)), \mathcal{A}(h_B(x))\}, \min\{\mathcal{A}(h_A(z)), \mathcal{A}(h_B(z))\}\} =$$
$$\min\{\mathcal{A}(h_{A\cap B}(z)), \mathcal{A}(h_{A\cap B}(z))\}.$$

Therefore, $A \cap B$ is also $\mathcal{A}$-convex. □

This proof is inspired by a similar result in [14]. However, in that case the definition of convexity was slightly different, and it is here adapted to the new definition.

Proposition 4 *Let X be an ordered space. If $\mathcal{A}(a, b) = \min\{a, b\}$ for all $a, b \in [0, 1]$, then $\mathcal{A}$ preserves convexity.*

Proof. Due to the definition of intersection it is clear that:

$$\min\{h_{A\cap B}(x)\} = \min\{h_A(x), h_B(x)\}, \forall x \in X.$$

As A is min-convex,

$$\min\{h_A(y)\} \geq \min\{\min\{h_A(x)\}, \min\{h_A(z)\}\}$$

and the same happens for B,

$$\min\{h_B(y)\} \geq \min\{\min\{h_B(x)\}, \min\{h_B(z)\}\}.$$

Let us check if $A \cap B$ fulfills our definition:

$$\min\{h_{A\cap B}(y)\} = \min\{h_A(y), h_B(y)\} \geq \min\{\min\{h_A(x)\},$$
$$\min\{h_A(z)\}, \min\{h_B(x)\}, \min\{h_B(z)\} = \min\{\min\{h_{A\cap B}(x)\}, \min\{h_{A\cap B}(z)\}\}$$

and we can see that $A \cap B$ is min-convex. □

3.2. The case under the minimum at some point

The second case is when there exist two points $\alpha_1, \alpha_2 \in [0, 1]$ such that $\mathcal{A}(\alpha_1, \alpha_2) < \min\{\alpha_1, \alpha_2\}$.

Proposition 5 *Let X be an ordered space. If $\mathcal{A}$ is an aggregation function such that there is at least one pair of mutually distinct elements $(\alpha_1, \alpha_2) \in [0, 1]^2$ for which $\mathcal{A}(\alpha_1, \alpha_2) < \min\{\alpha_1, \alpha_2\}$, then $\mathcal{A}$ does not preserve convexity.*

Proof. We are going to find a general counterexample, which can be used for any aggregation function assuming at least one value under the minimum. Let $\mathcal{A}$ be an aggregation function such that there are $\alpha_1, \alpha_2 \in [0, 1]$ with $\mathcal{A}(\alpha_1, \alpha_2) < \min\{\alpha_1, \alpha_2\}$.

Let us consider $X = \{x, y, z\}$ with $x < y < z$. Then we can define two hesitant fuzzy sets on X as:

$$h_A(x) = h_A(z) = \{\alpha_1, \alpha_2\}, h_A(y) = \{\mathcal{A}(\alpha_1, \alpha_2)\},$$

and

$$h_B(x) = h_B(y) = h_B(z) = \min\{\alpha_1, \alpha_2\}.$$

It is easy to check that both A and B are $\mathcal{A}$-convex. Their intersection is the hesitant fuzzy set defined by

$$h_{A\cap B}(x) = h_{A\cap B}(z) = \min\{\alpha_1, \alpha_2\}, h_{A\cap B}(y) = \{\mathcal{A}(\alpha_1, \alpha_2)\}.$$

Then

$$\mathcal{A}(h_{A\cap B}(y)) = \mathcal{A}(\alpha_1, \alpha_2) < \min\{\mathcal{A}(h_{A\cap B}(x)), \mathcal{A}(h_{A\cap B}(z))\} =$$

$$\min\{\min\{\alpha_1, \alpha_2\}, \min\{\alpha_1, \alpha_2\}\} = \min\{\alpha_1, \alpha_2\}.$$

So, the intersection is not $\mathcal{A}$-convex. □

This proof is illustrated in Figure 2, where we suppose that $\alpha_1 < \alpha_2$.

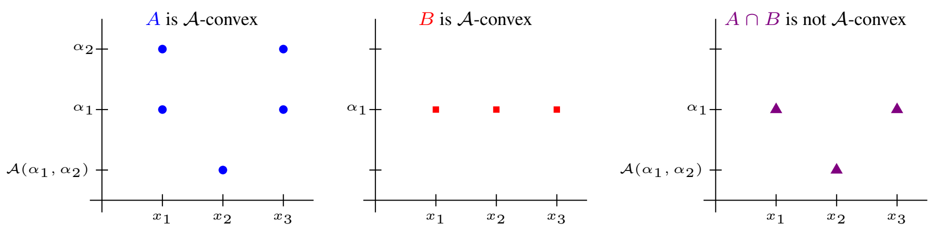


Fig. 2. Graphical proof of Proposition 5.

It is known that triangular norms (t-norms for short) are aggregation functions that fulfill $T(x, y) \leq \min\{x, y\}$ for all $(x, y) \in [0, 1]^2$. Thus, any t-norm different from the minimum fulfills that there exists a point $(\alpha_1, \alpha_2) \in [0, 1]^2$ such that $T(\alpha_1, \alpha_2) < \min\{\alpha_1, \alpha_2\}$. Therefore, by applying Proposition 5, any t-norm different from the minimum is not an appropriate choice for defining convexity, if we would expect convexity to be preserved under intersections.

3.3. The case over the maximum at some point

In the third case, we suppose that there exist points $\alpha_1, \alpha_2 \in [0, 1]$ such that the aggregation function fulfills $\mathcal{A}(\alpha_1, \alpha_2) > \max\{\alpha_1, \alpha_2\}$. The behavior with respect to the preservation of convexity under intersections and the way to prove is analogous to the previous case.

Proposition 6 *Let X be an ordered space. If $\mathcal{A}$ is an aggregation function such that there is at least one pair of mutually distinct elements $(\alpha_1, \alpha_2) \in [0, 1]^2$ for which $\mathcal{A}(\alpha_1, \alpha_2) > \max\{\alpha_1, \alpha_2\}$, then $\mathcal{A}$ does not preserve convexity.*

Proof. Now we are going to find a general counterexample, which can be used for any aggregation function under the conditions of the statement.

Let $X = \{x, y, z\}$ be the universe with $x < y < z$ and let A and B the hesitant fuzzy sets whose membership functions are:

$$h_A(x) = h_A(z) = \{\alpha_1, \alpha_2\}, h_A(y) = \{\mathcal{A}(\alpha_1, \alpha_2)\},$$

and

$$h_B(x) = h_B(y) = h_B(z) = \max\{\alpha_1, \alpha_2\}.$$

It is easy to check that both A and B are $\mathcal{A}$-convex. Their intersection is the hesitant fuzzy set given by

$$h_{A\cap B}(x) = h_{A\cap B}(z) = \{\alpha_1, \alpha_2\}, h_{A\cap B}(y) = \max\{\alpha_1, \alpha_2\}.$$

Then

$$\mathcal{A}(h_{A\cap B}(y)) = \mathcal{A}(\max\{\alpha_1, \alpha_2\}) = \max\{\alpha_1, \alpha_2\} <$$
$$\min\{\mathcal{A}(h_{A\cap B}(x)), \mathcal{A}(h_{A\cap B}(z))\} = \min\{\mathcal{A}(\alpha_1, \alpha_2), \mathcal{A}(\alpha_1, \alpha_2)\} =$$
$$\mathcal{A}(\alpha_1, \alpha_2) > \max\{h_{A\cap B}(x), h_{A\cap B}(z)\}.$$

So, the intersection is not $\mathcal{A}$-convex. □

This proof is again graphically illustrated. In this case in Figure 3, where we suppose that $\alpha_1 < \alpha_2$.

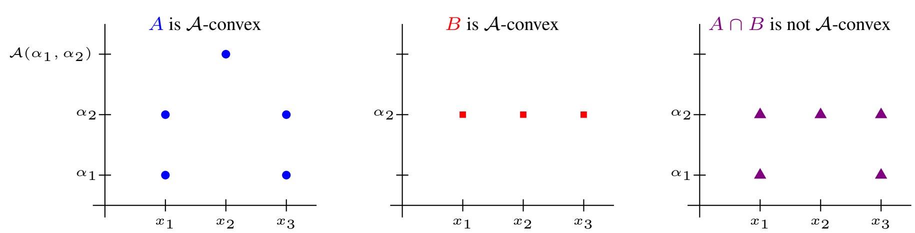


Fig. 3. Graphical proof of Proposition 6.

It is known that triangular conorms are aggregation functions that fulfill $S(x, y) \geq \max\{x, y\}$ for all $(x, y) \in [0, 1]^2$. Thus, any triangular conorm different from the maximum does not preserve convexity.

3.4. The case between minimum and maximum

Now we will study the only remaining case, aggregation functions which are between the minimum and the maximum, but they are not equal to any of them. In this case, we cannot describe the general behavior of this aggregation functions, since some of them preserve convexity for the intersection and some of them do not preserve it.

As we commented previously, the arithmetic mean is an aggregation function between minimum and maximum such that the intersection of two convex hesitant fuzzy sets need not be convex. This was already commented in [14] for their definition, but it is also true now for the new one, as we can see from the following example.

Example 2 *Let $X = \{x, y, z\}$ be the universe with $x < y < z$ and let A and B the hesitant fuzzy sets defined by:*

X	x	y	z
A	$\{0.4\}$	$\{0.2, 0.6\}$	$\{0.4\}$
B	$\{0.4\}$	$\{0.4\}$	$\{0.4\}$

It is clear that if $\mathcal{A}$ is the arithmetic mean, A and B are $\mathcal{A}$-convex. However, their intersection is defined as

X	x	y	z
$A \cap B$	$\{0.4\}$	$\{0.2, 0.4\}$	$\{0.4\}$

but $\mathcal{A}(y) = 0.3 < \min\{\mathcal{A}(x), \mathcal{A}(z)\} = 0.4$. *Thus,* $A \cap B$ *is not* $\mathcal{A}$*-convex.*

However, it is not true that any aggregation function in this family does not preserve convexity. We will introduce an example of an aggregation function in this family preserving convexity. First of all, we are going to consider a map on $[0,1]^2$.

This mapping is a function $\mathcal{A}_2 : [0,1]^2 \to [0,1]$ defined as follows

$$\mathcal{A}_2(x,y) = \begin{cases} \max\{x,y\} & \text{if } x,y \in [0,0.5] \\ \min\{x,y\} & \text{if } x,y \in (0.5,1] \\ 0.5 & \text{otherwise.} \end{cases}$$

Its graphical representation is given at Figure 4.

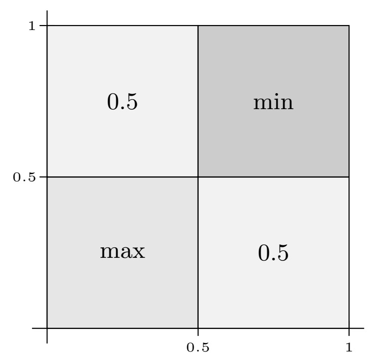


Fig. 4. Graphical representation of $\mathcal{A}_2$.

This mapping is a nullnorm (see [6]) and it is known that nullnorms are associative. So, it can be in a natural way extended to a mapping $\mathcal{A}_n : \cup_n [0,1]^n \to [0,1]$, which is also an aggregation function.

By definition it is also trivial that $\min\{\mathbf{x}\} \leq \mathcal{A}_n(\mathbf{x}) \leq \max\{\mathbf{x}\}$ for any $\mathbf{x} \in \cup_n [0,1]^n$. Moreover, $\mathcal{A}_2(0.7, 0.2) = 0.5$, so it is also clear that $\mathcal{A}_n$ is not equal in general to the minimum or the maximum.

Finally, we will see that $\mathcal{A}_n$-convexity is preserved by intersections.

Proposition 7 *The intersection of any two $\mathcal{A}_n$-convex hesitant fuzzy sets is a $\mathcal{A}_n$-convex hesitant fuzzy set.*

Proof. Let us suppose that X is an ordered space. Let A and B be $\mathcal{A}_n$-convex hesitant fuzzy sets. Let $x, y, z \in X$ such that $x < y < z$.

We will divide the proof into three cases:

1. The case $\mathcal{A}_n(h_{A\cap B}(y)) > 0.5$.
   (a) If $\mathcal{A}_n(h_{A\cap B}(x)) \leq 0.5$ or $\mathcal{A}_n(h_{A\cap B}(z)) \leq 0.5$ then

   $$\min\{\mathcal{A}_n(h_{A\cap B}(x)), \mathcal{A}_n(h_{A\cap B}(z))\} \leq 0.5 < \mathcal{A}_n(h_{A\cap B}(y))$$

   and therefore the condition to be $A \cap B$ $\mathcal{A}_n$-convex is fulfilled in this case.
   (b) If $\mathcal{A}_n(h_{A\cap B}(x)) > 0.5$ and $\mathcal{A}_n(h_{A\cap B}(z)) > 0.5$ then, by the definition of $\mathcal{A}_n$, $\mathcal{A}_n(h_{A\cap B}(x)) = \min\{h_{A\cap B}(x)\}$ and $\mathcal{A}_n(h_{A\cap B}(z)) = \min\{h_{A\cap B}(z)\}$ and considering the definition of the intersection (Definition 3), we have that $0.5 < \mathcal{A}_n(h_{A\cap B}(x)) = \min\{h_{A\cap B}(x)\} = \min\{h_A(x), h_B(x)\}$ and that $0.5 < \mathcal{A}_n(h_{A\cap B}(z)) = \min\{h_{A\cap B}(z)\} = \min\{h_A(z), h_B(z)\}$.
   Then,

   $$\min\{\mathcal{A}_n(h_{A\cap B}(x)), \mathcal{A}_n(h_{A\cap B}(z))\} = \min\{h_A(x), h_B(x), h_A(z), h_B(z)\} =$$

   $$\min\{\min\{\min\{h_A(x)\}, \min\{h_A(z)\}\}, \min\{\min\{h_B(x)\}, \min\{h_B(z)\}\}\}.$$

   As we noticed that $0.5 < \min\{h_A(x), h_B(x)\}$ and $0.5 < \min\{h_A(z), h_B(z)\}$, by definition of $\mathcal{A}_n$, we have that $\mathcal{A}_n(h_A(x)) = \min\{h_A(x)\}$, $\mathcal{A}_n(h_A(z)) = \min\{h_A(z)\}$, $\mathcal{A}_n(h_B(x)) = \min\{h_B(x)\}$ and $\mathcal{A}_n(h_B(z)) = \min\{h_B(z)\}$.
   Then,

   $$\min\{\mathcal{A}_n(h_{A\cap B}(x)), \mathcal{A}_n(h_{A\cap B}(z))\} =$$

   $$\min\{\min\{\mathcal{A}_n(h_A(x)), \mathcal{A}_n(h_A(z))\}, \min\{\mathcal{A}_n(h_B(x)), \mathcal{A}_n(h_B(z))\}\}.$$

   But, by the $\mathcal{A}_n$-convexity of A and B we have that

   $$\min\{\mathcal{A}_n(h_{A\cap B}(x)), \mathcal{A}_n(h_{A\cap B}(z))\} \leq \min\{\mathcal{A}_n(h_A(y)), \mathcal{A}_n(h_B(y))\}$$

   On the other hand, $\mathcal{A}_n(h_{A\cap B}(y)) > 0.5$, we also have that $\mathcal{A}_n(h_{A\cap B}(y)) = \min\{h_{A\cap B}(y)\} = \min\{h_A(y), h_B(y)\} = \min\{\min\{h_A(y)\}, \min\{h_B(y)\}\} = \min\{\mathcal{A}_n(h_A(y)), \mathcal{A}_n(h_B(y))\}$.
   Thus, we have proven that

   $$\min\{\mathcal{A}_n(h_{A\cap B}(x)), \mathcal{A}_n(h_{A\cap B}(z))\} \leq \mathcal{A}_n(h_{A\cap B}(y)).$$

2. If $\mathcal{A}_n(h_{A\cap B}(y)) < 0.5$, then $\mathcal{A}_n(h_{A\cap B}(y)) = \max\{h_{A\cap B}(y)\}$.
   By the definition of the intersection, we have $\max\{h_{A\cap B}(y)\} = \max\{h_A(y)\}$ or $\max\{h_{A\cap B}(y)\} = \max\{h_B(y)\}$. Suppose we have the first case (the proof for the second case it totally analogous). Since $\max\{h_A(y)\} < 0.5$, then $\mathcal{A}_n(h_A(y)) = \max\{h_A(y)\}$. By applying that A is a $\mathcal{A}_n$-convex hesitant fuzzy set, we have $\mathcal{A}_n(h_A(x)) \leq \mathcal{A}_n(h_A(y))$ or $\mathcal{A}_n(h_A(z)) \leq \mathcal{A}_n(h_A(y))$. Let us consider that we have the first case (again the second case is analogous). Thus, $\mathcal{A}_n(h_A(x)) < 0.5$ and then $\mathcal{A}_n(h_A(x)) = \max\{h_A(x)\} < 0.5$. By considering again the definition of the intersection, we see that $\max\{h_{A\cap B}(x)\} \leq \max\{h_A(x)\} < 0.5$ and therefore $\mathcal{A}_n(h_{A\cap B}(x)) = \max\{h_{A\cap B}(x)\}$. Now, if we join the above inequalities and equalities, we have:

   $$\mathcal{A}_n(h_{A\cap B}(x)) = \max\{h_{A\cap B}(x)\} \leq \max\{h_A(x)\} = \mathcal{A}_n(h_A(x)) \leq$$

   $$\mathcal{A}_n(h_A(y)) = \max\{h_{A\cap B}(y)\} = \mathcal{A}_n(h_{A\cap B}(y))$$

   and then

   $$\mathcal{A}_n(h_{A\cap B}(y)) \geq \min\{\mathcal{A}_n(h_{A\cap B}(x)), \mathcal{A}_n(h_{A\cap B}(z))\}.$$

3. The case $\mathcal{A}_n(h_{A\cap B}(y)) = 0.5$.
If $\mathcal{A}_n(h_{A\cap B}(x)) \leq 0.5$ or $\mathcal{A}_n(h_{A\cap B}(z)) \leq 0.5$, then the proof is trivial. Thus, we will consider that $\mathcal{A}_n(h_{A\cap B}(x)) > 0.5$ and $\mathcal{A}_n(h_{A\cap B}(z)) > 0.5$. In that case, $\mathcal{A}_n(h_{A\cap B}(x)) = \min\{h_{A\cap B}(x)\} = \min\{h_A(x) \cup h_B(x)\} > 0.5$ and $\mathcal{A}_n(h_{A\cap B}(z)) = \min\{h_{A\cap B}(z)\} = \min\{h_A(z) \cup h_B(z)\} > 0.5$. Then
$$\min\{h_A(x)\}, \min\{h_A(z)\}, \min\{h_B(x)\}, \min\{h_B(z)\} > 0.5$$
and therefore $\mathcal{A}_n(h_A(x)) = \min\{h_A(x)\} > 0.5$ and similarly we prove that
$$\mathcal{A}_n(h_A(z)), \mathcal{A}_n(h_B(x)), \mathcal{A}_n(h_B(z)) > 0.5.$$
As A and B are $\mathcal{A}_n$-convex, then $\mathcal{A}_n(h_A(y)) > 0.5$ and $\mathcal{A}_n(h_B(y)) > 0.5$. Then, $\min\{h_A(y) \cup h_B(y)\} > 0.5$ and therefore $\mathcal{A}_n(h_{A\cap B}(y)) = \min\{h_{A\cap B}(y)\} > 0.5$ which is a contradiction, so we can assure that $\mathcal{A}_n(h_{A\cap B}(x)) \leq 0.5$ or $\mathcal{A}_n(h_{A\cap B}(z)) \leq 0.5$ and therefore
$$\mathcal{A}_n(h_{A\cap B}(y)) = 0.5 \geq \min\{\mathcal{A}_n(h_{A\cap B}(x)), \mathcal{A}_n(h_{A\cap B}(z))\}.$$

Thus, we have proven that $A \cap B$ is $\mathcal{A}_n$-convex. □

Note that in fact any nullnorm and its extensions could be used in the previous demonstration.

As all the four cases are studied, we know the behavior of the different aggregation functions with respect to the preservation of convexity under intersections. Now we can say that only minimum, maximum and some specific aggregation functions between them are appropriate to define convexity for hesitant fuzzy sets.

Nowadays, there are many experiences where more than one attribute is to be combined into one overall value. In lots of cases, the attributes compensate for each other, to some extent. For teachers, there is a common situation that there are students with high motivation which compensate for less intelligence or vice versa [44]. Compensatory operators were introduced by Zimmermann and Zysno [43]. The main idea of these operators is to provide compensation between the small and large degrees of membership when combining fuzzy sets. Since then, several studies of these operators were presented [16,22,23]. For instance, Kolesarova and Komornikova gave the following definition in 1999:

Definition 10 *[17] An aggregation operator $\mathscr{A}$ on the unit interval is called a compensatory operator if for each n-tuple $(x_1, \ldots, x_n) \in [0,1]^n$, $n \in \mathbb{N}$, such that $\mathscr{A}(x_1, \ldots, x_n) \in]0,1[$ there exist elements $y, z \in [0,1]$ such that $\mathscr{A}(x_1, \ldots, x_n, y) < \mathscr{A}(x_1, \ldots, x_n) < \mathscr{A}(x_1, \ldots, x_n, z)$*

We can see with these comments that the idea of compensation is just to obtain low values when the argues are small and vice versa. For instance, the arithmetic mean is a compensatory operator which does not preserve convexity.

4. Illustrative example: optimization

As we mentioned in the introduction, the notion of convexity has been applied in many different contexts. One of the most interesting is in the field of optimization. A simple example of a possible application is shown here. We can use it to see the possible applications of the developed results.

Let us consider we have to design two new types of paint, which are prepared by mixing three primary elements: r, b and y. To obtain one bottle of the paint type A we need 3 units of r and 2 units of b. To obtain one bottle of the paint type B we need 5 units of r and 1 unit of y. The availability of r, b and y are 15 units, 6 units and 2 units, respectively. If at least one bottle has to be prepared, then the set of possible solutions about the number of bottles prepared for A and B is represented by dots in Figure 5.

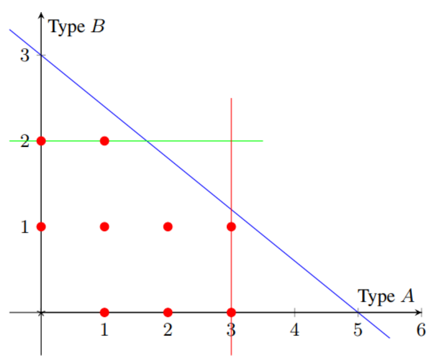


Fig. 5. Feasible region.

Depending on the number of bottles type A and B we are considering, we could obtain by different procedures, 9 different products, which will be represented by A_iB_j with $i = 0, 1, 2, 3$ and $j = 0, 1, 2$. If a committee of two experts evaluates the functionality and design of these products, we obtain two different hesitant fuzzy sets whose referential is the feasible region. Thus, let us suppose the values assigned to the functionality (F) and the design (D) for the different products are:

X	A_0B_1	A_0B_2	A_1B_0	A_1B_1	A_1B_2	A_2B_0	A_2B_1	A_3B_0	A_3B_1
F	$\{0.2, 0.6\}$	$\{0.2, 0.3\}$	$\{0.4, 0.6\}$	$\{0.5, 0.6\}$	$\{0.7, 0.6\}$	$\{0.6, 0.6\}$	$\{0.4, 0.5\}$	$\{0.2, 0.3\}$	$\{0.5, 0.1\}$
D	$\{0.4, 0.6\}$	$\{0.6, 0.5\}$	$\{0.6, 0.7\}$	$\{0.8, 0.7\}$	$\{0.9, 1\}$	$\{1, 1\}$	$\{0.9, 0.7\}$	$\{0.6, 0.7\}$	$\{0.5, 0.4\}$

If we consider the lexicographical order in $\mathbb{R}^2$, that is, $(x_1, y_1) \leq (x_2, y_2)$ iff $x_1 < x_1$ or $x_1 = x_2$ and $y_1 \leq y_2$, it is clear that F and D are min-convex hesitant fuzzy sets. Thus, the products fulfilling both properties could be represented as

X	A_0B_1	A_0B_2	A_1B_0	A_1B_1	A_1B_2	A_2B_0	A_2B_1	A_3B_0	A_3B_1
$F \cap D$	$\{0.2, 0.4, 0.6\}$	$\{0.2, 0.3\}$	$\{0.4, 0.6\}$	$\{0.5, 0.6\}$	$\{0.7, 0.6\}$	$\{0.6, 0.6\}$	$\{0.4, 0.5\}$	$\{0.2, 0.3\}$	$\{0.5, 0.1, 0.4\}$

By Proposition 4, the hesitant fuzzy set above, which represents the membership values to any product about both properties, is again a min-convex hesitant fuzzy sets. By

Proposition 2, any of its α-cut is a convex set. Thus, for a fixed level, we could optimize any objective function on this set by using the classical theorem about optimization. Of course, only in the case the aggregation function preserves the convexity under intersections, we could work in a similar way.

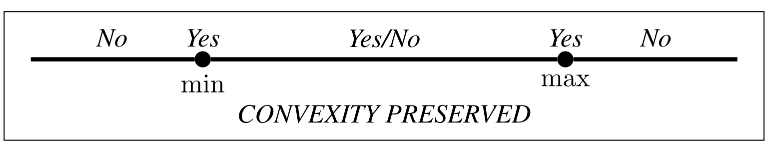
This easy example can be seen as a way to show the different possible applications of the obtained results for typical hesitant fuzzy sets. However, for a real application, some other structures could be considered in two senses. On one hand, the case the membership value is a multiset, to work in a different way if a value is considered just by one expert or by one hundred. On the other hand, the case of interval instead of discrete sets as membership values, i.e. interval-valued fuzzy sets, could be also necessary in some cases. However, typical hesitant fuzzy sets have been revealed as a very useful tool, as we commented on the introduction.

5. Conclusions

In this paper, we have introduced a new definition of convexity for typical hesitant fuzzy sets, which are finite hesitant fuzzy sets, based on aggregation functions. This definition can be considered for any ordered universe. Moreover, it coincides with the classical definition for fuzzy sets. It also has a good behavior with respect to the equivalence for cuts. About the preservation of the convexity under intersections, we have studied the behavior for various ways of aggregating values of a hesitant fuzzy set. Our results may be summarized as follows:

- Convexity is preserved for the minimum and the maximum.
- Convexity is not preserved if the aggregation function is under the minimum, respectively over the maximum, at some point.
- For the case between the minimum and the maximum, there are cases when convexity is preserved as well as those when it is not.

These results can also be visualized as follows:



The question of the exact characterization of the aggregation functions between minimum and maximum that preserve convexity remains to be open. So does the study of the preservation of the convexity for some possible general concepts of the intersection. This study can be seen as the first step in a general study. Thus, once we finish the theoretical part, we would like to combine it with some other extension of fuzzy sets, in order to apply it for some real problems about optimization under uncertainty.

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