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A Study on Optimally Constructed Compactly Supported Orthogonal Wavelet Filters

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Abstract. Compactly supported orthogonal wavelet filters are extensively applied to the analysis and description of abrupt signals in fields such as multimedia. Based on the application of an elementary method for compactly supported orthogonal wavelet filters and the construction of a system of nonlinear equations for filter coefficients, we design compactly supported orthogonal wavelet filters, in which both the scaling and wavelet functions have many vanishing moments, by approximately solving the system of nonlinear equations. However, when solving such a system about filter coefficients of compactly supported wavelets, the most widely used method, the Newton Iteration method, cannot converge to the solution if the selected initial value is not near the exact solution. For such, we propose optimization algorithms for the Gauss-Newton type method that expand the selection range of initial values. The proposed method is optimal and promising when compared to other works, by analyzing the experimental results obtained in terms of accuracy, iteration times, solution speed, and complexity.

Keywords: compactly supported orthogonal wavelets, the least-squares method, Gauss-Newton methods, LMF method.

1. Introduction

In signal analysis, modeling and processing, wavelets with compact support property and orthogonality are widely used. The advantages of wavelets with compact support property are: 1) when transforming truncated signals into finite-length signals, due to the infinite time-domain waveform length of compactly supported wavelet, the transformed signals are of finite length, i.e., there are no truncation errors; 2) FIR is the

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filter of compact-supported wavelets, which can make computation decreased to get favorable real-time property. Furthermore, orthogonal wavelets are useful to extract signal features for pattern recognition when applying in signal decomposition. The orthogonality reflected in digital image processing means that the original image's total energy is equal to the wavelet domain. So constructing compactly supported orthogonal wavelets plays an essential role in wavelet analysis.

Several researches to construct compactly supported orthogonal wavelets for general purposes were proposed, due to the several application advantages in multimedia technology. In [1], a framework for designing the compactly supported orthogonal wavelets in the time-domain was proposed, while an approach to design the compactly supported orthogonal wavelets from filter banks in poly-phase also having vanishing moments was proposed in [2]. Han et al. [3] constructed a family of compactly supported symmetric orthogonal complex wavelets with dilation 4 and the shortest possible supports to their orders of vanishing moments. Hiroshi Toda et al. [4] proposed a new type of orthonormal wavelet basis having customizable frequency bands. Its frequency bands can be freely designed with arbitrary bounds in the frequency domain. In [5, 6], a series of compactly supported orthogonal wavelet bases with various features were constructed based on the structure of orthogonal conjugate filters.

There are also some works for matched compactly supported orthogonal wavelets. A design for matched wavelets and matched scaling function was proposed in [7], which was done in the time-domain and did not have assumptions on the template function. In [8], the problem of designing the compactly supported orthogonal wavelets for finite-length signals was firstly addressed. In [9], a novel approach was presented to design orthogonal wavelets matched to a signal with compact support and vanishing moments. It provided a systematic and versatile framework for matching an orthogonal wavelet to a specific signal or application.

Many researchers were interested in the study of constructing compactly supported orthogonal wavelets with B-splines. He, T. and T. Nguyen[10] gave an approach to prove Daubechies' result on the existence of spline type orthogonal scaling functions and to evaluate Daubechies scaling functions. Tung Nguyen[11] presented a method to construct orthogonal spline-type scaling functions by using B-spline functions. To induce the orthogonality and remain property of compact support, a class of polynomial function factors to the B-splines' masks was multiplied. In [12], Yang Shouzhi *et al.* used orthonormalization procedure so that splines become orthogonal scaling functions. Then to make them have the property of compact support, the weighted average method was used to eliminate the denominator of the two-scale symbol. Gupta, K.L. et al. developed a simple procedure to generate the compactly supported orthogonal scaling function for higher-order B-splines and multiplied the mask of B-spline with a polynomial function that satisfied all the conditions as the mask of B-spline to obtain orthogonality [13].

Wavelet transform, as a useful analytic tool for unstable signals, is also widely applied to artificial intelligence, such as computer classification and recognition, artificial synthesis of music and language, medical imaging and diagnosis and so on. Especially, wavelet transform performs well in boundary processing and multi-scale edge detection. Some researchers are devoted to computer vision field. [14] used UNet method to preprocessing under the guidance of medical knowledge. Then, multi-scale receiving field convolution module was used to extract features of the segmented images with different sizes. [15] proposed a novel click-boosted graph ranking framework for image retrieval, which consists of two coupled components. [16] used an adopted TextRank to extract key sentences and a template-based method to construct questions from key sentences. Then a multi-feature neural network model was built for ranking to obtain the top questions.

Wavelet transform is not only used in signal and image processing, speech recognition and synthesis, but also in digital image encryption algorithms. In the field of digital copyright protection, how to encrypt and protect digital information often involves the encryption and concealment of digital images and their watermark information, as well as the encryption and concealment of video and audio and their watermark information. In terms of the confirmation of digital rights, ledgers and whole-process records are open in blockchain. Digital copyright works are distributedly stored in network nodes by providing decentralized distributed technology. Now blockchain plays a significant role in digital copyright protection, but there are also some limitations of blockchain in the aspects of privacy protection, data processing and data storage. There are many researchers devoted to the study of privacy protection and data security based on cloud computing and blockchain. The enhanced secure access schemes for outsourced and a secure deduplication scheme proposed in [17-20] can effectively protect local data. [21] proposed one secure data integrity verification scheme for cloud storage. [22-24] proposed a secure blockchain-based schemes for IoT data credibility, a secure data storage and recovery in industrial blockchain network environments and a blockchain-based scheme to protect data confidentiality and traceability.

Except for the case of order one, B-splines of orders greater than one are not orthogonal. Methods that can make wavelets obtain orthogonality must be applied to construct compactly supported orthogonal wavelet and scaling filters. However, when constructing the compactly supported orthogonal wavelets for general purposes, applying other methods to get orthogonality makes construction complicated. Therefore, we construct the compactly supported orthogonal wavelet filters by solving nonlinear equations about filter coefficients with optimization methods and obtaining a series of wavelet filters with different linear phases and multiresolution properties.

We construct compactly supported orthogonal wavelet filters based on an elementary method in [25]. Bi-scale equation and the similarity between orthogonality condition and the formula $\left(cos^2\frac{\omega}{2}+sin^2\frac{\omega}{2}\right)^2=1$ are used to construct a system of nonlinear equations about filter coefficients of compactly supported orthogonal wavelets. We derive a system of nonlinear equations about filter coefficients using double-scale equation and properties of wavelet and scaling filters. Most of the existing methods for solving the system of nonlinear equations are Newton Iteration Method and its improved forms. However, Newton Iteration Method has a fatal shortcoming -- dependence of the solution on the initial values. That is, if the initial values are not correctly selected, likely, the solution of the system of nonlinear equations will not be obtained. The least-square method can expand the selection range of initial values, and sometimes for the same initial values, the Newton iteration method may not converge to the correct solution, but the least square method can [26]. Based on the Gauss-Newton method for the least-square problem, we propose optimization algorithms to solve the system of nonlinear equations. When the given initial values of filter coefficients vary, solutions to the system of nonlinear equations are different. Thus, we can obtain a series of compactly supported orthogonal wavelets with various features.

Our contributions in constructing compactly supported orthogonal wavelet filters are:

1. deriving system of nonlinear equations about filter coefficients of compactly supported orthogonal wavelets. We derive a system of nonlinear equations about filter coefficients in the case of L=1, L=2 and L=3, then we can obtain compactly supported orthogonal wavelets filters with the length of 4, 6 and 8;
2. proposing optimization algorithms to solve a system of nonlinear equations derived. Based on the Gauss-Newton type method's traditional algorithms, we propose optimization algorithms: 1) Basic Gauss-Newton method; 2) Damping Gauss-Newton method under Wolfe criterion; 3) Gauss-Newton method with QR decomposition; 4) LMF-Dogleg method that adds Dogleg method for trust-region subproblem into LMF method.

In terms of accuracy, iteration times, and complexity of algorithms, we analyze approximate solution results of equations to draw conclusions. We analyze solutions obtained by algorithms of Basic GS method, Damping GS method, GS method with QR decomposition, LMF method, and LMF-Dogleg method to obtain the conclusion. We conclude that the compactly supported orthogonal wavelet bases obtained by the basic Gauss-Newton method are optimal.

The remaining of this article is structured as follows. Section 2 introduces the related concepts and properties of wavelet, the elementary method for constructing compactly supported wavelet filters, and related algorithms for solving the least-square problem. Systems of nonlinear equations about filters coefficients are derived in section3. Section 4 proposes optimization algorithms based on the Gauss-Newton method, and approximate solution results and analysis are presented in section5, and finally, concluding remarks and future directions are given in Section 6.

2. Preliminaries

We present: 1) related concepts and properties of wavelets; 2) an elementary method in [25] is used to construct compactly supported orthogonal wavelet filters in section3; 3) traditional algorithms of Gauss-Newton type method for the least-square problem are used to propose optimization algorithms for solving a system of nonlinear equations in section4.

2.1. Related Concepts and Properties

The following concepts, properties, and theorems are used to construct compactly supported orthogonal wavelet filters in section3.

Definition 1. (Two-scale Equation)

$$\varphi(t) = \sum_n h_n \varphi_{1,n}(t) = \sqrt{2} \sum_n h_n \, \varphi(2t - n) \tag{1}$$

$$\psi(t) = \sum_n g_n \varphi_{1,n}(t) = \sqrt{2} \sum_n g_n\, \varphi(2t - n) \tag{2}$$

Equations (1) and (2) are named two-scale equation $\varphi(t)$ and $\psi(t)$ are standard orthogonal basis functions in scale space V_0 and wavelet space W_0 respectively, and the expansion coefficients are

$$h_n = <\varphi(t), \varphi_{1,n}(t)>$$

$$g_n = <\psi(t), \varphi_{1,n}(t)>$$

The two-scale equation describes the intrinsic and essential relationship between the basis functions of two adjacent scale spaces V_{j-1} and V_j, or adjacent space V and wavelet space W.

Definition 2. (Low-pass Filter)

$$H(\omega) = \frac{1}{\sqrt{2}} \sum_n h_n e^{-in\omega} \tag{3}$$

$\{h_n\}$ is the corresponding coefficient of low-pass filter.

Definition 3. (High-pass Filter)

$$\mathbf{G}(\boldsymbol{\omega}) = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}} \sum_{\mathbf{n}} \mathbf{g_n} \mathbf{e}^{-\mathbf{in}\boldsymbol{\omega}} \tag{4}$$

$\{g_n\}$ is the corresponding coefficient of high-pass filter.

Theorem 1. $\boldsymbol{H(\omega)}$ and $\boldsymbol{G(\omega)}$ are $\boldsymbol{2\pi}$-periodic functions, and satisfy

$$|H(\omega)|^2 + |H(\omega + \pi)|^2 = 1 \tag{5}$$

$$|G(\omega)|^2 + |G(\omega + \pi)|^2 = 1 \tag{6}$$

Theorem 2.

1)

$$\sum_n h_n = \sqrt{2}, \sum_n g_n = 0 \tag{7}$$

2)

$$\sum_n (-1)^n h_n = 0 \tag{8}$$

3)

$$\sum_{k\in Z} h_{2k} = \sum_{k\in Z} h_{2k+1} = \frac{\sqrt{2}}{2} \quad (9)$$

4)

$$\sum_{k\in Z} |h_k|^2 = 1 \quad (10)$$

2.2. An Elementary Method for Constructing Compactly Supported Orthogonal Wavelets

We construct compactly supported orthogonal wavelets based on an elementary proposed in [25]. Bi-scale equation and the similarity between orthogonality condition and the formula $\left(cos^2\frac{\omega}{2} + sin^2\frac{\omega}{2}\right)^2 = 1$ were used to construct a system of nonlinear equations about filter coefficients of compactly supported orthogonal wavelets. The process of construction is as follows:

Let

$$H(\omega) = \frac{1}{\sqrt{2}}\sum_{n=0}^{M} h_n e^{-in\omega},$$

$$H(0) = 1, |H(\omega)|^2 + |H(\omega+\pi)|^2 = 1,$$

and for the properties of $H(\omega)$, there is

$$|H(\omega)|^2 = \frac{1}{2}\sum_{k=0}^{M} h_k^2 + \sum_{n=1}^{M}(\sum_{j=0}^{M-n} h_j h_{j+n})\, cos\, n\omega = \sum_{n=0}^{M} \zeta_n\, cos\, n\omega,$$

namely

$$|H(\omega)|^2 = \sum_{n=0}^{M} \zeta_n\, cos\, n\omega$$

$$\zeta_0 = \frac{1}{2}\sum_{k=0}^{M} h_k^2\, , \zeta_n = \sum_{j=0}^{M-n} h_j h_{j+n}\, , n = 1,2, \dots, M. \quad (11)$$

To calculate $\{h_n\}$ using formula 11, values of $|H(\omega)|^2$ or $\{\zeta_n\}$ must be given first. In [25], $\{h_n\}$ is calculated by giving the values of $\{\zeta_n\}$, and values of $\{\zeta_n\}$ are derived by using similarity between $\left(cos^2\frac{\omega}{2} + sin^2\frac{\omega}{2}\right)^2 = 1$ and the orthogonality condition $|H(\omega)|^2 + |H(\omega+\pi)|^2 = 1$. Generally, let

$$|H(\omega)|^2 + |H(\omega+\pi)|^2 = 1 = \left(cos^2\frac{\omega}{2} + sin^2\frac{\omega}{2}\right)^{2L}.$$

When L is the general case,

$$|H(\omega)|^2 + |H(\omega+\pi)|^2 = 1 = \left(cos^2\frac{\omega}{2} + sin^2\frac{\omega}{2}\right)^{2L}$$

$$= \sum_{n=0}^{2L} \binom{2L}{n} (cos^2\frac{\omega}{2})^{2L-n} (sin^2\frac{\omega}{2})^n$$

$$= \left(cos^2\frac{\omega}{2}\right)^{2L} + \sum_{n=1}^{L-1} \binom{2L}{n} \left(cos^2\frac{\omega}{2}\right)^{2L-n} \left(sin^2\frac{\omega}{2}\right)^n + \binom{2L}{L} (cos^2\frac{\omega}{2})^L (sin^2\frac{\omega}{2})^L + \sum_{n=L+1}^{2L-1} \binom{2L}{n} \left(cos^2\frac{\omega}{2}\right)^{2L-n} \left(sin^2\frac{\omega}{2}\right)^n + \left(sin^2\frac{\omega}{2}\right)^{2L}$$

$$= (cos^2\frac{\omega}{2})^{2L} + \sum_{n=1}^{L-1} \binom{2L}{n} \left(cos^2\frac{\omega}{2}\right)^{2L-n} \left(sin^2\frac{\omega}{2}\right)^n + \binom{2L}{L} (cos^2\frac{\omega}{2})^{L+1} (sin^2\frac{\omega}{2})^L + \binom{2L}{L} (cos^2\frac{\omega}{2})^L (sin^2\frac{\omega}{2})^{L+1} + \sum_{n=L+1}^{2L-1} \binom{2L}{n} \left(cos^2\frac{\omega}{2}\right)^{2L-n} \left(sin^2\frac{\omega}{2}\right)^n + \left(sin^2\frac{\omega}{2}\right)^{2L}.$$

Let

$$|H(\omega)|^2 = (cos^2\frac{\omega}{2})^{2L} + \sum_{n=1}^{L-1} \binom{2L}{n} \left(cos^2\frac{\omega}{2}\right)^{2L-n} \left(sin^2\frac{\omega}{2}\right)^n + \binom{2L}{L} (cos^2\frac{\omega}{2})^{L+1} (sin^2\frac{\omega}{2})^L, \quad \textbf{(12)}$$

$$|H(\omega+\pi)|^2 = \left(sin^2\frac{\omega}{2}\right)^{2L} + \sum_{n=L+1}^{2L-1} \binom{2L}{n} \left(cos^2\frac{\omega}{2}\right)^{2L-n} \left(sin^2\frac{\omega}{2}\right)^n + \binom{2L}{L} (cos^2\frac{\omega}{2})^L (sin^2\frac{\omega}{2})^{L+1}. \quad \textbf{(13)}$$

2.3. The Least-Squares Problem (LS)[27]

Definition 4. Giving a set of experimental data $(\boldsymbol{t_i}, \boldsymbol{y_i})(\boldsymbol{i} = \mathbf{1}, \dots, \boldsymbol{m})$ and a functional model $\boldsymbol{f}(\boldsymbol{x}; \boldsymbol{t})$, remaining quantity $\boldsymbol{r_i}(\boldsymbol{x})$ is $\boldsymbol{r_i}(\boldsymbol{x}) = \boldsymbol{y_i} - \boldsymbol{f}(\boldsymbol{x}; \boldsymbol{t_i}), \quad \boldsymbol{i} = \mathbf{1}, \dots, \boldsymbol{m}$. The least-squares problem is

$$\min f(x) = \frac{1}{2}\sum_{i=1}^{m} r_i^2\,(x) = \frac{1}{2} r(x)^T r(x), \quad x\epsilon\mathbb{R}^n, \quad m \geq n. \tag{14}$$

Definition 5. The Newton equation for solving the least-squares problem is

$$(J_k^T J_k + S_k) d_k = -J_k^T r_k, \tag{15}$$

where

$$J(x) = [\nabla r_1(x), \dots, \nabla r_m(x)]^T \in \mathbb{R}^{m\times n},$$

$$S(x) = \sum_{i=1}^{m} r_i(x) \nabla^2 r_i(x), \tag{16}$$

$$J_k = J(x_k), S_k = S(x_k).$$

The methods of solving the least-squares problem can also be used to get approximate solution $x = [x_1, x_2, \dots, x_n]^T$ of a system of nonlinear equations

$$r(x) = [r_1(x), \dots, r_m(x)]^T = 0.$$

2.4. Gauss-Newton Method

Definition 6. Gauss-Newton Equation

$$J_k^T J_k d_k = -J_k^T r_k. \tag{17}$$

The advantage of the Gauss-Newton method to solve the system of nonlinear equations lies is that it does not need to calculate the second derivative of $r(x)$[27]. So the minimum point d_k in the linear least-squares problem with respect to d can be obtained by computing the value of $d_k = -(J_k^T J_k)^{-1} J_k^T r_k$.

The following is the traditional algorithm of the Gauss-Newton method:

Algorithm 1. Gauss-Newton Method to Solve LS[27]
Step 1. Give $x_0, \varepsilon > 0, k = 0$;
Step 2. if the termination condition is satisfied, stop the iteration;
Step 3. solve $J_k^T J_k d = -J_k^T r_k$ to obtain d_k ;
Step 4. compute $x_{k+1} = x_k + \alpha_k d_k$, $k = k + 1$, go back to Step 2.

2.5. QR Decomposition of $\mathbf{J_k}$

To reduce the solution sensitivity caused by the rounding error in solving LS with the basic Gauss-Newton method, and improve the feasibility of the solution process and accuracy of the final solution, in [27], QR decomposition of J_k were used to obtain d_k. Solving the Gauss-Newton equation is equivalent to solve

$$min \frac{1}{2} \parallel Q_k^T J_k d + Q_k^T r_k \parallel^2.$$

First, QR decomposition is used to obtain

$$J_k = Q_k \begin{bmatrix} R_k \\ 0 \end{bmatrix}, \quad \textbf{(18)}$$

where $Q_k \in \mathbb{R}^{m \times m}$ is an orthogonal matrix, $R_k \in \mathbb{R}^{n \times n}$ is the upper triangular matrix with non-zero diagonal elements and $0 \in \mathbb{R}^{(m-n) \times n}$ is a zero matrix.

Then partition Q_k to get

$$Q_k = \begin{bmatrix} Q_1^{(k)} & Q_2^{(k)} \end{bmatrix}, \quad \textbf{(19)}$$

where $Q_1^{(k)} \in \mathbb{R}^{m \times n}$, $Q_2^{(k)} \in \mathbb{R}^{m \times (m-n)}$.

And let

$$Q_k^T r_k = \begin{bmatrix} Q_1^{(k)^T} \\ Q_2^{(k)^T} \end{bmatrix} r_k = \begin{bmatrix} b1 \\ b2 \end{bmatrix}_{m-n}^{n}, \quad \textbf{(20)}$$

so

$$\begin{aligned} \parallel J_k d + r_k \parallel^2 &= \parallel Q_k^T J_k d + Q_k^T r_k \parallel^2 \\ &= \parallel \begin{bmatrix} R_k \\ 0 \end{bmatrix} d + Q_k^T r_k \parallel^2 \quad \textbf{(21)} \\ &= \parallel R_k d + b_1 \parallel^2 + \parallel b_2 \parallel^2. \end{aligned}$$

d_k is the solution of the least-squares problem if and only if d_k is the solution of $R_k d = -b_1$.

2.6. LMF Method

Definition 7. LMF Equation

$$(J_k^T J_k + v_k I) d = -J_k^T r_k \quad \textbf{(22)}$$

To solve the situation that $J_k^T J_k$ is singular in the iteration process with the Gauss-Newton method, Levenberg proposed the LMF equation to obtain d_k, where $v_k \geq 0$. LM method is a trust region method, and the value of v_k can be modified in iterations with the idea of the trust region method. LMF method solves the least-squares problem with the following algorithm:

Algorithm 2. LMF Method to Solve LS [16]

Step 1. Give $x_0 \in \mathbb{R}^n, v_0 > 0, \varepsilon > 0, k = 0$;
Step 2. if the termination condition $\| g(x_k) \| \leq \varepsilon$ is satisfied, stop the iteration;
Step 3. solve $(J_k^T J_k + v_k I)d = -J_k^T r_k$ to obtain d_k;
Step 4. compute r_k with $r_k = \frac{\Delta f_k}{\Delta q_k}$;
Step 5. if $r_k < 0.25$, $v_{k+1} = 4v_k$; else if $r_k > 0.75$, $v_{k+1} = \frac{v_k}{2}$; else $v_{k+1} = v_k$;
Step 6. if $r_k \leq 0$, $x_{k+1} = x_k$; else $x_{k+1} = x_k + d_k$, $k = k + 1$ go back to Step 2.

And

$$\Delta f_k = f(x_k) - f(x_k + d_k),$$

$$\Delta q_k = q_k(0) - q_k(d_k) = \frac{1}{2} d_k^T (v_k d_k - g_k), \tag{23}$$

$$g_k = J_k^T r_k.$$

2.7. Dogleg Method

Powell proposed the Dogleg method to solve the trust-region subproblem in the LMF method, since the direction d_k^{LM} obtained from the LM equation is affected by the value of v_k, which makes solution speedy varied greatly. The algorithm of the Dogleg method is:

Algorithm 3. Dogleg Method to Solve Trust Region Subproblem

Step 1. Give $\Delta_k > 0, J_k, r_k$;
Step 2. if $\| d_k^{GN} \| \leq \Delta_k$, $d_k = d_k^{GN}$, and output d_k, stop the iteration;
Step 3. compute $\alpha_k = \frac{\| d_k^{SD} \|^2}{\| J_k d_k^{SD} \|^2}$; if $\alpha_k \| d_k^{SD} \| \geq \Delta_k$,
$d_k = \frac{\Delta_k}{\| d_k^{SD} \|} d_k^{SD}$, and output d_k;
Step 4. solve the unary quadratic equation
$\| d_k^{GN} - \alpha_k d_k^{SD} \|^2 \beta^2 + 2\alpha_k {d_k^{SD}}^T (d_k^{GN} - \alpha_k d_k^{SD})\beta + \alpha_k^2 \| d_k^{SD} \|^2 - \Delta_k^2$ to obtain β (take the solution greater than 0) ;
Step 5. compute $d_k = (1 - \beta)\alpha_k d_k^{SD} + \beta d_k^{GN}$, and output d_k, stop the iteration.

And d_k^{GN} is Gauss-Newton direction, i.e.

$$d_k^{GN} = -(J_k^T J_k)^{-1} J_k^T r_k, \quad d_k^{SD} = -J_k^T r_k. \tag{24}$$

3. Construction of a system of nonlinear equations

Based on the method presented in subsection 2.2, we derive a system of nonlinear equations about filter coefficients of compactly supported wavelets in the case of L=1, L=2 and L=3 to obtain filters with the length of 4, 6, and 8. In subsection 2.2, $|H(\omega)|^2$ can be presented as

$$|H(\omega)|^2 = (cos^2\frac{\omega}{2})^{2L} + \sum_{n=1}^{L-1}\binom{2L}{n}\left(cos^2\frac{\omega}{2}\right)^{2L-n}\left(sin^2\frac{\omega}{2}\right)^n + \binom{2L}{L}(cos^2\frac{\omega}{2})^{L+1}(sin^2\frac{\omega}{2})^L.$$

When the value of L is given, $|H(\omega)|^2$ can be presented as

$$|H(\omega)|^2 = \sum_{n=0}^{M}\zeta_n\cos n\omega$$

to obtain values of $\{\zeta_n\}$, which are used to construct a system of nonlinear equations about $\{h_n\}$. Derivations of values of $\{\zeta_n\}$ in the case of L=1, L=2, and L=3 are as follows:

3.1. L=1

When L=1, let

$$\begin{aligned}|H(\omega)|^2 &= cos^4\frac{\omega}{2} + 2cos^4\frac{\omega}{2}sin^2\frac{\omega}{2} = \left(\frac{1+cos\omega}{2}\right)^2 + \frac{1}{4}sin^2\omega(1+cos\omega)\\
&= \frac{1}{4}(1+2cos\omega+cos^2\omega+sin^2\omega+cos\omega sin^2\omega)\\
&= \frac{1}{4}\left[2+2cos\omega+\frac{1}{4}(cos\omega sin^2\omega+cos\omega sin^2\omega)\right]\\
&= \frac{1}{2}+\frac{1}{2}cos\omega+\frac{1}{16}[cos\omega(1-cos2\omega)+sin\omega sin2\omega]\\
&= \frac{1}{2}+\frac{1}{2}cos\omega+\frac{1}{16}(cos\omega-cos\omega cos2\omega+sin\omega sin2\omega)\\
&= \frac{1}{2}+\frac{9}{16}cos\omega-\frac{1}{16}cos3\omega\end{aligned}$$

then we have values of $\{\zeta_n\}$ in the case of L=1,

$$\{\zeta_0,\zeta_1,\zeta_2,\zeta_3\} = \left\{\frac{1}{2},\frac{9}{16},0,-\frac{1}{16}\right\}. \quad \textbf{(25)}$$

3.2. L=2

When L=2, let

$$|H(\omega)|^2 = cos^8\frac{\omega}{2} + 4cos^6\frac{\omega}{2}sin^2\frac{\omega}{2} + 6cos^6\frac{\omega}{2}sin^4\frac{\omega}{2}$$

$$= \left(\frac{1+cos\omega}{2}\right)^4 + 4\left(\frac{1+cos\omega}{2}\right)^2 sin^2\frac{\omega}{2}cos^2\frac{\omega}{2} + 6cos^2\frac{\omega}{2}cos^4\frac{\omega}{2}sin^4\frac{\omega}{2}$$

$$= \frac{1}{16}[(1+cos\omega)^4 + 4(1+cos\omega)^2 sin^2\omega + 3(1+cos\omega)sin^4\omega]$$

$$= \frac{1}{16}(cos^4\omega + 4cos^3\omega + 6cos^2\omega + 4cos\omega + 1 + 4cos^2\omega sin^2\omega + 8cos\omega sin^2\omega + 4sin^2\omega + 3cos\omega sin^4\omega + 3sin^4\omega)$$

$$= \frac{1}{16}(1 + 4cos\omega + cos^4\omega + cos^2\omega sin^2\omega + 3cos^2\omega sin^2\omega + 3sin^4\omega + 4cos^3\omega + 6cos^2\omega + 4sin^2\omega + 8cos\omega sin^2\omega + 3cos\omega sin^4\omega)$$

$$= \frac{1}{16}(1 + 4cos\omega + 7cos^2\omega + 7sin^2\omega + 4cos\omega + 4cos\omega sin^2\omega + 3cos\omega sin^4\omega)$$

$$= \frac{1}{16}(8 + 8cos\omega + 4cos\omega sin^2\omega + 3cos\omega sin^4\omega)$$

$$= \frac{1}{2} + \frac{1}{16}\left(8cos\omega - \frac{25}{16}cos3\omega + \frac{3}{16}cos5\omega + \frac{22}{16}cos\omega\right)$$

$$= \frac{1}{2} + \frac{75}{128}cos\omega - \frac{25}{256}cos3\omega + \frac{3}{256}cos5\omega,$$

then we have values of $\{\zeta_n\}$ in the case of L=2,

$$\{\zeta_0, \zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5\} = \left\{\frac{1}{2}, \frac{75}{128}, 0, -\frac{25}{256}, 0, \frac{3}{256}\right\}. \quad \textbf{(26)}$$

3.3. L=3

When L=3, let

$$|H(\omega)|^2 = (cos^2\frac{\omega}{2})^6 + \sum_{n=1}^{2}\binom{6}{n}(cos^2\frac{\omega}{2})^{6-n}(sin^2\frac{\omega}{2})^n + \binom{6}{3}(cos^2\frac{\omega}{2})^4(sin^2\frac{\omega}{2})^3$$

$$= cos^{12}\frac{\omega}{2} + 6cos^{10}\frac{\omega}{2}sin^2\frac{\omega}{2} + 15cos^8\frac{\omega}{2}sin^4\frac{\omega}{2} + 20cos^8\frac{\omega}{2}sin^6\frac{\omega}{2}$$

$$= (\frac{1+cos\omega}{2})^6 + 6(\frac{1+cos\omega}{2})^4cos^2\frac{\omega}{2}sin^2\frac{\omega}{2} + 15(\frac{1+cos\omega}{2})^2cos^4\frac{\omega}{2}sin^4\frac{\omega}{2} + 20(\frac{1+\cos\omega}{2})cos^6\frac{\omega}{2}sin^6\frac{\omega}{2}$$

$$= \frac{1}{64}[(1+cos\omega)^6 + 6(1+cos\omega)^4sin^2\omega + 15(1+cos\omega)^2sin^4\omega + 10(1+cos\omega)sin^6\omega]$$

$$= \frac{1}{64}\left[\sum_{k=0}^{6}\binom{6}{k}(cos\omega)^{6-k} + 6sin^2\omega\sum_{k=0}^{4}\binom{4}{k}(cos\omega)^{4-k} + 15(1+2cos\omega+cos^2\omega)sin^4\omega + 10cos\omega sin^6\omega + 10sin^6\omega\right]$$

$$= \frac{1}{64}(1 + 6cos\omega + 15cos^2\omega + 6sin^2\omega + 20cos^3\omega + 24cos\omega sin^2\omega + 15cos^4\omega + 15sin^4\omega + 36cos^2\omega sin^2\omega + 6cos^5\omega + 30cos\omega sin^4\omega + 24cos^3\omega sin^2\omega + cos^6\omega + 10sin^6\omega + 6cos^4\omega sin^2\omega + 15cos^2\omega sin^4\omega + 10cos\omega sin^6\omega)$$

$$= \frac{1}{64}(7 + 26cos\omega + 24cos^2\omega + 15sin^2\omega + 6cos^3\omega + 22cos\omega sin^2\omega + cos^4\omega + 10sin^4\omega + 11cos^2\omega sin^2\omega + 12cos\omega sin^4\omega + 10cos\omega sin^6\omega)$$

$$= \frac{1}{64}(22 + 32cos\omega + 10cos^2\omega + 10sin^2\omega + 16cos\omega sin^2\omega + 12cos\omega sin^4\omega + 10cos\omega sin^6\omega)$$

$$= \frac{1}{64}(32 + 32cos\omega + 16cos\omega sin^2\omega + 12cos\omega sin^4\omega + 10cos\omega sin^6\omega)$$

$$= \frac{1}{2} + \frac{1}{64}\left(\frac{1225}{32}cos\omega - \frac{245}{32}cos3\omega + \frac{49}{32}cos5\omega - \frac{5}{32}cos7\omega\right)$$

$$= \frac{1}{2} + \frac{1225}{2048}cos\omega - \frac{245}{2048}cos3\omega + \frac{49}{2048}cos5\omega - \frac{5}{2048}cos7\omega,$$

then we have values of $\{\zeta_n\}$ in the case of L=3,

$$\{\zeta_0,\zeta_1,\zeta_2,\zeta_3,\zeta_4,\zeta_5,\zeta_6,\zeta_7\}=\left\{\frac{1}{2},\frac{1225}{2048},0,-\frac{245}{2048},0,\frac{49}{2048},0,-\frac{5}{2048}\right\}. \quad \textbf{(27)}$$

Then use formula (11) and theorem 2.2 to conduct a system of nonlinear equations about filter coefficients $\{h_n\}$ in the case of L=1, L=2 and L=3 as follows:

when L=1,

$$\begin{cases} h_0^2+h_1^2+h_2^2+h_3^2=1 \\ h_0h_1+h_1h_2+h_2h_3=\frac{9}{16} \\ h_0h_2+h_1h_3=0 \\ h_0h_3=-\frac{1}{16} \\ h_0+h_1+h_2+h_3=\sqrt{2} \end{cases}$$

when L=2,

$$\begin{cases} h_0^2+h_1^2+h_2^2+h_3^2+h_4^2+h_5^2=1 \\ h_0h_1+h_1h_2+h_2h_3+h_3h_4+h_4h_5=\frac{75}{128} \\ h_0h_2+h_1h_3+h_2h_4+h_3h_5=0 \\ h_0h_3+h_1h_4+h_2h_5=-\frac{25}{256} \\ h_0h_4+h_1h_5=0 \\ h_0h_5=\frac{3}{256} \\ h_0+h_1+h_2+h_3+h_4+h_5=\sqrt{2} \end{cases}$$

when L=3,

$$\begin{cases} h_0^2+h_1^2+h_2^2+h_3^2+h_4^2+h_5^2+h_6^2+h_7^2=1 \\ h_0h_1+h_1h_2+h_2h_3+h_3h_4+h_4h_5+h_5h_6+h_6h_7=\frac{1225}{2048} \\ h_0h_2+h_1h_3+h_2h_4+h_3h_5+h_5h_7=0 \\ h_0h_3+h_1h_4+h_2h_5+h_3h_6+h_4h_7=-\frac{245}{2048} \\ h_0h_4+h_1h_5+h_2h_6+h_3h_7=0 \\ h_0h_5+h_1h_6+h_2h_7=\frac{49}{2048} \\ h_0h_6+h_1h_7=0 \\ h_0h_7=-\frac{5}{2048} \\ h_0+h_1+h_2+h_3+h_4+h_5+h_6+h_7=\sqrt{2} \end{cases}$$

Compactly supported orthogonal wavelet filters with the length of 4, 6, and 8 can be obtained by approximately solving the abovementioned three systems of nonlinear equations with filter coefficients $\{h_n\}$. In the following section, we propose optimization algorithms for the approximate solution of the system of nonlinear equations.

4. Optimization Algorithms

To obtain optimal compactly supported orthogonal wavelet bases by solving the system of nonlinear equations constructed in section3, we propose optimization algorithms of variants of the Gauss-Newton method. When the dimension of the system of nonlinear equations is small, the Gauss-Newton type method can locally converge to the solution of a system of nonlinear equations fast. Since the upper dimension of the nonlinear system constructed in section 3 is only 8, we propose the Basic Gauss-Newton method, Damping Gauss-Newton method and QR decomposition Method based on GN method, and Dogleg method based on LMF method.

4.1. Basic Gauss-Newton Method

The basic Gauss-Newton method refers to the Gauss-Newton method on $\alpha_k = 1$, as its most significant advantage, is that the algorithm is simple and easy to be implemented. The basic Gauss-Newton method solves the least-squares problem with the following algorithm:

Algorithm 4. Basic Gauss-Newton Method to Solve LS

Step 1. Give $x_0, \varepsilon > 0, k = 0$;
Step 2. if the termination condition $\| g(x_k) \| \leq \varepsilon$ is satisfied, stop the iteration;
Step 3. solve $J_k^T J_k d = -J_k^T r_k$ to obtain d_k ;
Step 4. compute $x_{k+1} = x_k + d_k, \quad k = k + 1,$ go back to Step 2.

4.2. Damping Gauss-Newton Method.

The damping Gauss-Newton method refers to the Gauss-Newton method with line search. We use the inexact line search under the Wolfe criterion to obtain α_k, namely let $\beta \epsilon (0,1),\ \sigma \epsilon (0,0.5),\ \alpha_k = \beta^{m_k}$, where m_k is the minimal nonnegative integer satisfying the following inequality:

$$f(x_k + \beta^m d_k) \leq f(x_k) + \sigma \beta^m g_k^T d_k. \qquad \textbf{(28)}$$

Compared with the basic Gauss-Newton method, the damping Gauss-Newton method has higher precision due to line search to find the step size and is relatively more complex. The algorithm of the damping Gauss-Newton method to solve LS is as follows:

Algorithm 5. Damping Gauss-Newton Method to Solve LS

Step 1. Give $x_0, \varepsilon > 0, k = 0$;
Step 2. if the termination condition $\| g(x_k) \| \leq \varepsilon$ is satisfied, stop the iteration;
Step 3. solve $J_k^T J_k d = -J_k^T r_k$ to obtain d_k;
Step 4. give $\beta \epsilon (0,1)$, $\sigma \epsilon (0,0.5)$, m_k=0;
Step 5. if the current m_k satisfies $f(x_k + \beta^m d_k) \leq f(x_k) + \sigma \beta^m g_k^T d_k$, stop the iteration;
Step 6. or else $m = m + 1$, go back to Step5;
Step 7. compute $\alpha_k = \beta^{m_k}$;
Step 8. compute $x_{k+1} = x_k + \alpha_k d_k$, $k = k + 1$, go back to Step 2.

4.3. Gauss-Newton Method with QR Decomposition.

Based on QR decomposition of J_k, d_k can be obtained by solving the equation $R_k d = -b_1$, which reduces the solution sensitivity caused by the rounding error in solving LS with the Gauss-Newton method, and improves the feasibility of the solution process and the accuracy of the final solution. The QR decomposition iteration method is used to solve the least-square problem with the following algorithm:

Algorithm 6. Gauss-Newton Method with QR Decomposition to Solve LS

Step 1. Give $x_0, \varepsilon > 0, k = 0$;
Step 2. if the termination condition $\| g(x_k) \| \leq \varepsilon$ is satisfied, stop the iteration;
Step 3. compute the QR decomposition of J_k;
Step 4. compute $b_1 = {Q_1^{(k)}}^T r_k$;
Step 5. solve the upper triangular equation $R_k d = -b_1$ to obtain d_k;
Step 6. compute $x_{k+1} = x_k + d_k$, $k = k + 1$, go back to Step 2.

4.4. LMF-Dogleg Method

Based on the LMF method, we propose the LMF-Dogleg method that uses Dogleg method to solve the subproblem of selecting d_k in the LMF method. LMF-Dogleg method is used to solve the least-square problem with the following algorithm:

Algorithm 7. LMF-Dogleg Method to Solve LS

Step 1. Give $x_0 \in \mathbb{R}^n, \Delta_k > 0, \varepsilon > 0, k = 0$;
Step 2. if the termination condition $\| g(x_k) \| \leq \varepsilon$ is satisfied, stop the iteration;
Step 3. if $\| d_k^{GN} \| \leq \Delta_k$, $d_k = d_k^{GN}$, and output d_k;
Step 4. compute $\alpha_k = \frac{\| d_k^{SD} \|^2}{\| J_k d_k^{SD} \|^2}$; if $\alpha_k \| d_k^{SD} \| \geq \Delta_k$ and $d_k = \frac{\Delta_k}{\| d_k^{SD} \|} d_k^{SD}$,output d_k;
Step 5. solve the unary quadratic equation

$\| d_k^{GN} - \alpha_k d_k^{SD} \|^2 \beta^2 + 2\alpha_k {d_k^{SD}}^T (d_k^{GN} - \alpha_k d_k^{SD})\beta + \alpha_k^2 \| d_k^{SD} \|^2 - \Delta_k^2$ to obtain β （take the solution greater than 0）;

Step 6. compute $d_k = (1-\beta)\alpha_k d_k^{SD} + \beta d_k^{GN}$, and output d_k;

Step 7. update Δ_k: compute $r_k = \frac{\Delta f_k}{\Delta q_k}$; if $r_k < 0.25$, $\Delta_{k+1} = \Delta_k/4$; else if $r_k > 0.75$, $\Delta_{k+1} = 2\Delta_k$; $\Delta_{k+1} = \Delta_k$;

Step 8. if $r_k \leq 0$, $x_{k+1} = x_k$; else $x_{k+1} = x_k + d_k$, and $k = k+1$, go back to step 2.

5. Analysis

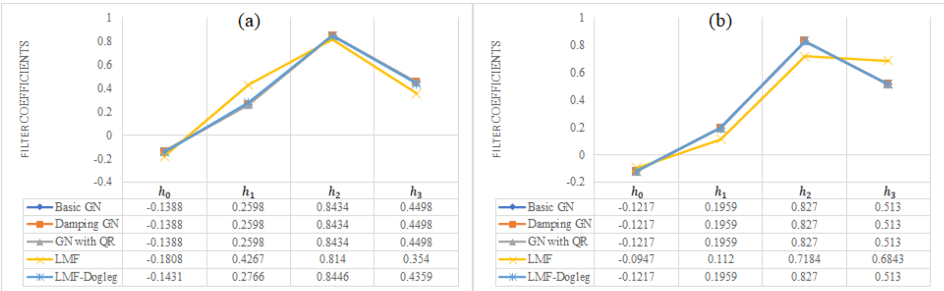
To obtain the optimal compactly supported orthogonal wavelet bases with lengths of 4, 6, and 8, we implement the optimization algorithms proposed in section 4 by MATLAB to solve the system of nonlinear equations derived in section3. Giving two sets of initial values of $\{h_n\}$, we obtain different solutions of the three systems of nonlinear equations. Then we analyze the approximate solution results to obtain the best compactly supported orthogonal wavelet bases constructed based on the elementary method proposed in [8]. We remain to solve precision $\varepsilon = 0.001$, trust region radius $\Delta_0 = 2$, and $v_0 = 10$ unchanged, and only make the initial values of $\{h_n\}$ vary. The initial values and solutions are presented as follows:

1. When L=1, let the initial values $h_1^{(0)} = \{0,1,1,1\}$ and $\{0,0,1,1\}$ respectively. The solving results are presented in Fig.1, which gives two sets of graphs corresponding to different initial values and explicitly indicate the solution's structure. Furthermore, the data of two tables in Fig.1 are presented to analyze the accuracy of algorithms.
2. When L=2, let the initial values $h_2^{(0)} = \{1,0,1,1,1,1\}$ and $\{0,0,0,1,1,1\}$ respectively. The solving results are presented in Fig.2, where it is plain that five curves of solution are almost coincident.
3. When L=3, let the initial values $h_3^{(0)} = \{1,0,1,1,1,1,1,1\}$ and $\{0,0,0,0,1,1,1,1\}$ respectively. The solving results are presented in Fig.3. It can be seen from the two graphs that solution curves of different algorithms are almost coincident except for the LMF. The detailed analyses about this phenomenon are discussed as follows.

The selection of initial values and optimization algorithms affects the accuracy and solving speed of approximate solutions, as can be seen in Fig.1, Fig.2, and Fig.3 that the structure of the solution varies with algorithms and initial values. Therefore, we analyze the solution results and figure out under what circumstances the method can be applied to obtain optimal compactly support orthogonal wavelet bases. We analyze the solution results on the accuracy, iteration times (of different cases presented in Table1), solution speed, and algorithms' complexity. The accuracy of the approximate solutions is analyzed based on the degree of coincidence between the solution results and the constant terms in the original equation group (the checking method is to substitute the solution results into the left side of the system of nonlinear equations and compare the values obtained with the right side).

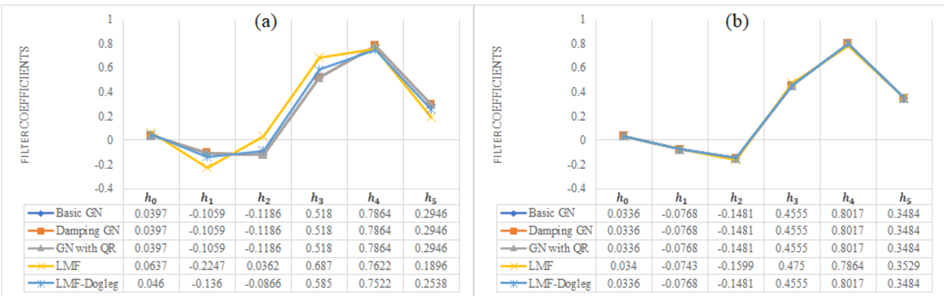
Table 1. Data in the table are iterations of different solving processes in Fig.1, Fig.2, and Fig.3, given by MATLAB programs. They are collected to measure the solving speed and complexity of different algorithms

Initial Values / Algorithms	L=1.(a)	L=1.(b)	L=2.(a)	L=2.(b)	L=3.(a)	L=3.(b)
Basic GN	8	6	11	5	8	8
Damping GN	8	6	11	5	11	8
GN with QR	8	6	11	5	8	8
LMF	9	6	10	7	18	11
LMF-Dogleg	8	6	10	5	7	10



L = 1

Fig.1. To analyze the influence of variations about initial values of filter coefficients on approximate solution results. Among the initial values, solving precision $\varepsilon = 0.001$, trust-region radius $\Delta_0 = 2$, and $v_0 = 10$ remain unchanged, and only the initial filter coefficients are variables. The initial values of filter coefficients of (a) are $\{0,1,1,1\}$, and $\{0,0,1,1\}$ of (b)



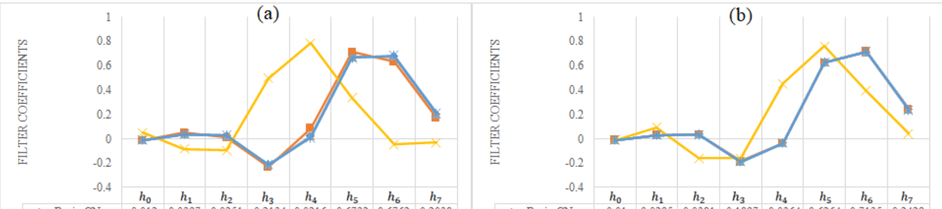
L = 2

Fig.2. Solving precision $\varepsilon = 0.001$, trust region radius $\Delta_0 = 2$, and $v_0 = 10$ remain unchanged. The initial values of filter coefficients of (a) are $\{1,0,1,1,1,1\}$, and $\{0,0,0,1,1,1\}$ of (b). It can be seen in (b) that five curves of solution are almost coincident, and compared with (a), the structure of solution in (a) are basically tallies (b)

Analysis of the solution speed is based on the solution process's iteration times, which is presented in Table 1 (given by the MATLAB program). The iteration times are the number of iterations used until the termination condition $\| g(x_k) \| \leq \varepsilon$ is satisfied.

The solution results of (a) in each figure show that when using various methods to approximate the solution, the difference between the structure obtained and the iteration coefficient is significant; the results of (b) in Fig.1, Fig.2, and Fig.3 relatively show a small or even no difference between most of them. Accordingly, the results in the chart(a) in each figure serve as group1, and results in chart(b) serve as group2 to analyze the accuracy and solve the speed of different algorithms.

The features of initial values selected in the group1 are: 1) there is only one 0 element; 2) the rest elements are all 1. We can see from Fig.1, Fig.2, and Fig.3 that iteration times of LMF are more than other algorithms: 1) when L=1 and 2, the number of iterations of the Gauss-Newton method is less than or equal to the LMF-Dogleg method; 2) when L=3, iterations times of the LMF-Dogleg solution is lower than the Gauss-Newton method. After checking the results in group1, we obtain conclusions: 1) when L=1 and 2, the accuracy of the Gauss-Newton method is better than that of the LMF method and LMF-Dogleg method; 2) when L=3, LMF-Dogleg method, the LMF method, and damping Gauss-Newton method are similar in accuracy and higher than the basic Gauss-Newton method. In addition, the structure of approximate solution results of the LMF method is different from that of other methods.



(a)	h_0	h_1	h_2	h_3	h_4	h_5	h_6	h_7
Basic GN	-0.012	0.0397	0.0251	-0.2134	0.0216	0.6732	0.6762	0.2038
Damping GN	-0.0141	0.0516	0.0078	-0.2342	0.0876	0.7106	0.6324	0.1725
GN with QR	-0.012	0.0397	0.0251	-0.2134	0.0216	0.6732	0.6762	0.2038
LMF	0.0502	-0.0819	-0.0955	0.4969	0.7864	0.3365	-0.0453	-0.0328
LMF-Dogleg	-0.0114	0.0365	0.0312	-0.2129	0.0112	0.6629	0.6839	0.2131

(b)	h_0	h_1	h_2	h_3	h_4	h_5	h_6	h_7
Basic GN	-0.01	0.0295	0.0381	-0.1897	-0.0364	0.6264	0.7135	0.2429
Damping GN	-0.0102	0.0305	0.035	-0.1856	-0.0372	0.6239	0.7179	0.2399
GN with QR	-0.01	0.0295	0.0381	-0.1897	-0.0364	0.6264	0.7135	0.2429
LMF	-0.0098	0.0964	-0.159	-0.162	0.4531	0.7609	0.3926	0.0423
LMF-Dogleg	-0.0104	0.0315	0.0334	-0.1866	-0.033	0.6273	0.7162	0.2357

L = 3

Fig.3. The initial values of filter coefficients of (a) are {1,0,1,1,1,1,1,1} and {0,0,0,0,1,1,1,1} of (b). Solving precision $\varepsilon = 0.001$, trust region radius $\Delta_0 = 2$, and $v_0 = 10$ remain unchanged. It can be seen in the two charts that curves of solutions are almost coincident except for the LMF

The feature of initial values selected in the group2 is that half of the elements of $\{h_n\}$ are 0, and the others are all 1. The solution results show that the accuracy of the group2 is higher than group1. Therefore, the initial values selected in group2 can be regarded as close to the exact solution. We can see from these three tables: 1) When L=1 and 2, only the results of LMF methods are different from others, and the number of iterations is higher or equal to others; 2) When L=3, the relation of iteration times is: LMF > LMF-Dogleg > Gauss-Newton type method. The checking results show that in three cases, the LMF method has the lowest accuracy; 3) When L=3, the LMF-Dogleg method and damping Gauss-Newton method are more accurate than other methods, which are similar to group1.

Synthesizing the above analysis, we can see that although the differences between initial values have effects on the solution results, in most cases, the algorithms in this paper can converge to the solution when approximately solving the system of nonlinear equations. After checking, we find that these effects are not noticeable: the difference of precision is only 0.001, and the difference of iteration times on L=1 and L=3 is minimal.

In addition, compared with the Newton Iteration method, it is more efficient to solve the system of nonlinear equations about filter coefficients by the least square method, and the solution process is relatively stable and efficient. Moreover, the efficiency of the LMF method is inferior in all aspects, and as with the improved LMF method, the LMF-Dogleg is superior to the LMF method in all aspects of analysis. The Gauss-Newton method was superior to the LMF-Dogleg method when L=1 and L=2, and slightly inferior to the LMF-Dogleg method and Gauss-Newton method when L=3, but the difference between the three methods was very small. In addition, the LMF-Dogleg method and damping Gauss-Newton method are more complicated than the basic Gauss-Newton method, and in the process of the actual execution of the program, the cost of time and space is more than the basic Gauss-Newton method. However, the precision and iterative coefficients are almost identical. We can see the basic Gauss-Newton method is easy to implement and has high precision. Therefore, we can conclude that the wavelet filter bases that are constructed by solving a system of nonlinear equations about filter coefficients with the basic Gauss-Newton method are optimum.

6. Conclusions and Future Work

We construct compactly supported orthogonal wavelet filters with different linear phases and multiresolution properties by solving the systems of nonlinear equations about filter coefficients. We drive these systems of nonlinear equations about filter coefficients of wavelets using bi-scale equation and similarity between orthogonality condition and the formula $\left(cos^2\frac{\omega}{2}+sin^2\frac{\omega}{2}\right)^2=1$. To approximately solve these systems of nonlinear equations, we propose optimization algorithms of the Basic Gauss-Newton method, Damping Gauss-Newton method, Gauss-Newton method with QR decomposition, and LMF-Dogleg method. We then analyze the solution results by these algorithms on the accuracy, iteration times, solution speed, and complexity of algorithms and conclude that the wavelet filters are constructed by a solving system of nonlinear equations about filter coefficients with basic Gauss-Newton method are optimal.

The general form of a system of nonlinear equations was not be given in this paper. As the derivation has reached 16 power when L=4, the construction is too complicated, and we cannot guarantee the correctness of the nonlinear system's constant term. Therefore, we only construct compactly supported wavelet filters in the cases of L=1, L=2, and L=3. Nevertheless, we obtain some laws :

1. There are 2L+2 unknown numbers and 2L+3 equations in the system of the nonlinear equation;
2. The general simplified result is: $|H(\omega)|^2=\frac{1}{2}+\sum_{k=0}^{L}(-1)^L\alpha_k\cos(2k+1)\omega$, where α_k is a coefficient.

As future work, we will attempt to find the general laws between the filter coefficients, and construct a system of nonlinear equations about them, then design a general program that can output the filter coefficients only by giving the value of L. In this way, we can improve the efficiency of constructing compactly supported orthogonal wavelet filters with various features, which are widely applied to signal analysis and processing in multimedia and other fields.

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