

Complete meet-continuous codomain lattice and extremal solutions of L -valued fuzzy relation equations*

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Abstract. We prove the existence of a maximal solution to any lattice-valued fuzzy relational equation, inequation, or system, in which infimum, supremum, and the standard composition are used. This result holds under the assumption that, in the codomain lattice, the infimum operation commutes with suprema of chains. To establish this result, we first recall and extend several lattice-theoretic properties under the assumption of meet-continuity. In particular, we recall that meet-continuity of the codomain lattice ensures composition-continuity in the lattice of L -valued relations. Building on these properties, we introduce algorithms for constructing the least solution to a broad class of equations and inequations, and prove that these algorithms terminate after at most countably many steps.

Keywords: Meet-Continuity, Lattice-Valued Relations, Relational Equations, Relational Inequations.

1. Introduction

Meet-continuity was defined implicitly in the works of Tarski on algebraic logic, closure operators, and complete lattices [23,24], while the very term meet-continuity appeared later, in the works of Keimel, Gierz, Hofmann, Keimel, Lawson, Mislove, Scott, etc. [8,9,13]

In complete lattices, the condition of being a meet-continuous is weaker than infinite codistributivity. In the latter case, lattice operators $\phi_a(x) = a \wedge x$ commute with any supremum (of possibly infinitely many elements), while in the case of a meet-continuous lattice, such operators commute only with the suprema of chains. For a finite chain, such a condition is superficial, since it holds in any lattice. So, the point is that operators of the type $\phi_a(x) = a \wedge x$ commute with infinite chains. However, not only does this condition fail to imply infinite codistributivity, but it also fails to imply even the ordinary

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codistributivity (which is satisfied in distributive lattices). Every finite non-distributive lattice shows this, since every finite lattice is meet-continuous. Take, for example, the following Pentagon lattice: $L = \{0, 1, a, b, c\}$, where $0, 1, a, b, c$ are any elements in the given relationship (Figure 1). This lattice is meet-continuous, but it is not distributive (and hence neither infinite-distributive as well.) Indeed, we have: $a \wedge (b \vee c) \neq (a \wedge b) \vee (a \wedge c)$, since $a \wedge (b \vee c) \neq a \wedge 1 = a$ and $(a \wedge b) \vee (a \wedge c) = 0 \vee 0 = 0$ (Figure 1). This lattice can be a co-domain lattice in this framework.

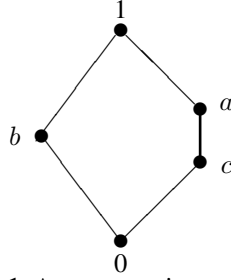


Fig. 1. A meet-continuous, non-distributive lattice.

While infinite codistributivity implies meet-continuity, finite codistributivity (which holds in distributive lattices) does not. The lattice in Figure 2 is distributive, but not meet-continuous, since $a = a \wedge \bigvee_{n \in \mathbb{N}} b_n > \bigvee_{n \in \mathbb{N}} (a \wedge b_n) = 0$

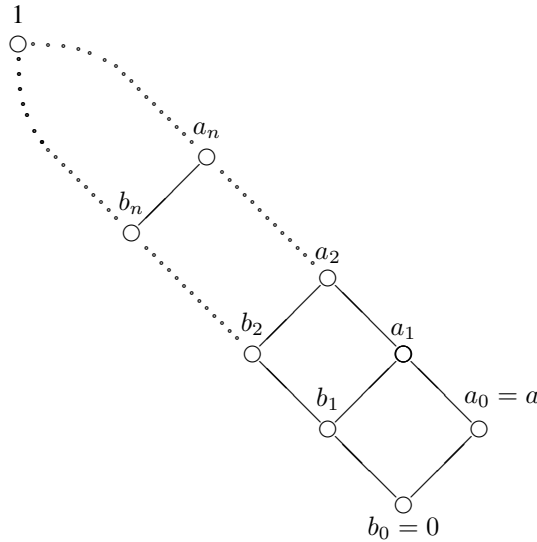


Fig. 2. A distributive lattice which is not meet-continuous.

In this manuscript, we are considering lattice-valued relational equations and inequations with a complete meet-continuous codomain lattice. We find extremal solutions of such equations and inequations. We use the fact that the meet-continuity of the codomain lattice implies composition-continuity in the lattice of L-valued relations [20]. We introduce a sort of general lattice-valued relational equation and inequation, using the usually

defined composition, as well as the infimum and supremum. In order to prove the existence of a maximal solution to the general equation, inequation, and their systems, we prove that any term with n variables - considered as a mapping from the lattice of n -tuples of relations to the lattice of fuzzy relations - commutes with the supremum of chains of n -tuples of fuzzy relations.

The significance of these results in general algebra and its applications also lies in the fact that all algebraic lattices, such as the lattice of all subalgebras or the lattice of all congruences (i.e., equivalences compatible with all algebra operations) of an algebra, are meet-continuous (see [25]). The lattice of weak congruences (symmetric and transitive relations compatible with all algebra operations) - a lattice that contains the congruence lattices of all the subalgebras - is also an algebraic lattice, thus it is meet-continuous. Thus, the property of algebraicity and consequent meet-continuity of these well-known lattices constitute a part of the motivation to consider the fuzzy sets, relations, and their (in)equations in the case of a meet-continuous codomain lattice.

2. Related work

Goguen introduced L -valued sets, relations, and their composition [10], thereby generalizing the notions of fuzzy sets, fuzzy relations, and their composition. Building on this framework, various special cases have been studied in order to obtain more refined results concerning fuzzy relations and their systems. These include approaches based on additional conditions, such as infinite distributivity [11,17], as well as extensions of the underlying structure through additional operations, for example, in the setting of residuated lattices [3,6,14,16,18].

Infinite distributivity of infimum over supremum, or multiplication over supremum (in case of residual codomain lattice) implies the existence of the greatest solution to some lattice-valued relational inequations or equations [18,22] (sometimes under additional condition of solvability). But infinite distributivity is a very strong condition in this context. Therefore, in 2018, the existence of a maximal solution to some lattice-valued relational inequations and equations was proved under a weaker condition of meet-continuity of the codomain lattice [19]. These results were extended to more types of equations in [20] due to the fact that the meet-continuity of the codomain lattice implies what we call composition-continuity in the lattice of L -valued relations. In this manuscript, we prove that, actually, these two continuities are equivalent.

In 2022, the existence of the greatest and the least solution was proved for some lattice-valued relational equations and inequations, in the general case of a complete codomain lattice [21]. Here we provide algorithms for the construction of the least solution of a wide class of inequations, equations, and their systems, in the case of a meet-continuous codomain lattice. They produce the least solutions in at most ω steps, where ω is the first (i.e., the least) countable ordinal, thus also proving the existence of the least solution in these cases.

3. Preliminaries

A sufficient condition for having a maximal element in a non-empty poset is given in the following well-known lemma from set theory:

Lemma 1. [4](Zorn) *If in a non-empty poset every chain has an upper bound, there exists at least one maximal element in the poset.*

Let A be a set, and L a lattice. An L -valued set on A (or L -valued subset of A) is any mapping $\mu : A \rightarrow L$. In the most general case A can be any set, but in applications, A is usually finite or countable.

Set of all L -valued sets on A is denoted by L^A ;

L^A is a lattice, relative to the following ordering:

$$\mu \leq \nu \Leftrightarrow (\forall a \in A)(\mu(a) \leq \nu(a))$$

We say that the lattice L is the codomain lattice of L^A .

An L -valued relation on A is any L -valued set on A^2 , i.e. any mapping $P : A^2 \rightarrow L$. Thus, $L^{A \times A}$ is the set of all L -valued relations on A .

If L is a complete lattice, $P, Q \in L^{A \times A}$, the composition of fuzzy relations is usually defined by:

$$(P \circ Q)(x, y) = \bigvee_{z \in A} (P(x, z) \wedge Q(z, y))$$

Note that this composition is an order-preserving operation in $L^{A \times A}$, relative to both operands.

The completeness of a codomain lattice is transferred to the lattice L^A . It is easy to check that supremum and infimum relative to the above componentwise ordering are also componentwise, i.e., that:

$$\begin{aligned} \left(\bigvee_{i \in I} \mu_i \right)(a) &= \bigvee_{i \in I} \mu_i(a) \\ \left(\bigwedge_{i \in I} \mu_i \right)(a) &= \bigwedge_{i \in I} \mu_i(a) \end{aligned}$$

Definition 1. *An element a of a complete lattice L is said to be meet-continuous if the following holds:*

$$a \wedge \bigvee_{i \in I} x_i = \bigvee_{i \in I} (a \wedge x_i)$$

for every chain $\{x_i \mid i \in I\} \subseteq L$.

A complete lattice L is said to be meet-continuous if all its elements are meet-continuous.

The property of being meet-continuous is transferred from the codomain lattice to the set of fuzzy sets:

Proposition 1. [20] *If A is a set and L a complete meet-continuous lattice, the poset L^A of fuzzy sets on A with membership values in L is also a complete meet-continuous lattice.*

This proposition also applies to the lattice $L^{A \times A}$ of fuzzy relations on A with membership values in L , since it may be seen as the lattice of fuzzy sets on $A \times A$ with membership values in L .

A more special notion of algebraicity of a complete lattice is introduced using the notion of compactness of a lattice element.

Definition 2. An element l of a complete lattice L is compact if the following holds: for every $U \subseteq L$, if $l \leq \bigvee U$, there exist a finite subset F of U such that $l \leq \bigvee F$.

Definition 3. A lattice L is algebraic if every $l \in L$ is the supremum of compact elements less than or equal to l .

Proposition 2. [25] Every algebraic lattice is meet-continuous.

Meet-continuity of the codomain lattice is connected to so-called composition-continuity of the lattice $L^{A \times A}$. Since the composition is not generally commutative, we define both left and right composition-continuity:

Definition 4. The lattice $L^{A \times A}$ is left composition-continuous if and only if

$$P \circ (\bigvee_{i \in I} Q_i) = \bigvee_{i \in I} (P \circ Q_i)$$

whenever $P \in L^{A \times A}$, and $\{Q_i \mid i \in I\}$ is a chain of fuzzy relations from $L^{A \times A}$

Definition 5. The lattice $L^{A \times A}$ is right composition-continuous if and only if

$$(\bigvee_{i \in I} Q_i) \circ P = \bigvee_{i \in I} (Q_i \circ P),$$

whenever $P \in L^{A \times A}$, and $\{Q_i \mid i \in I\}$ is a chain of fuzzy relations from $L^{A \times A}$

Both left and right composition-continuity are fulfilled if and only if the following property is also fulfilled:

Definition 6. The lattice $L^{A \times A}$ is composition-continuous if and only if

$$(\bigvee_{i \in I} P_i) \circ (\bigvee_{i \in I} Q_i) = \bigvee_{i \in I} (P_i \circ Q_i)$$

whenever $\{(P_i, Q_i) \mid i \in I\}$ is a chain of pairs of fuzzy relations from $L^{A \times A}$ relative to the componentwise order $((P_i, Q_i) \leq (P_j, Q_j) \text{ iff } P_i \leq P_j \text{ and } Q_i \leq Q_j)$

Lemma 2. [20] If A is a set, L a complete, meet-continuous lattice and $L^{A \times A}$ the set of L -valued relations on A , then $L^{A \times A}$ is both left and right composition-continuous. Moreover, it is composition-continuous.

4. Results

4.1. Some lattice-theoretic properties

We generalize the notion of meet-continuity (and composition-continuity as well):

Definition 7. Let L be a complete lattice and $\star : L \times L \rightarrow L$ an order-preserving binary operation in L . An element $a \in L$ is said to be left $\star$ -continuous if for any chain $\{l_i \mid i \in I\} \subseteq L$ we have:

$$a \star (\bigvee_{i \in I} l_i) = \bigvee_{i \in I} (a \star l_i)$$

If this holds for all $a \in L$, we say that L is left $\star$ -continuous.

Definition 8. Let L be a complete lattice and $\star : L \times L \rightarrow L$ an order-preserving binary operation in L . An element $a \in L$ is said to be right $\star$ -continuous if for any chain $\{l_i \mid i \in I\} \subseteq L$ we have:

$$(\bigvee_{i \in I} l_i) \star a = \bigvee_{i \in I} (l_i \star a)$$

If this holds for all $a \in L$, we say that L is right $\star$ -continuous.

Definition 9. Let L be a complete lattice and $\star : L \times L \rightarrow L$ an order-preserving binary operation in L . An element $a \in L$ is said to be $\star$ -continuous if it is both left and right-continuous.

If this holds for all $a \in L$, we say that L is $\star$ -continuous.

Lemma 3. L is a $\star$ -continuous lattice if and only if for every two chains in L $\{x_i \mid i \in I\}$ and $\{y_i \mid i \in I\}$, we have the following:

$$\left(\bigvee_{i \in I} x_i\right) \star \left(\bigvee_{i \in I} y_i\right) = \bigvee_{i \in I} (x_i \star y_i)$$

Proof. Since by monotony of $\star$ we have:

$$\begin{aligned} x_i \star y_i &\leq \left(\bigvee_{i \in I} x_i\right) \star \left(\bigvee_{i \in I} y_i\right), \text{ we have that:} \\ \left(\bigvee_{i \in I} x_i\right) \star \left(\bigvee_{i \in I} y_i\right) &\geq \bigvee_{i \in I} (x_i \star y_i) \end{aligned}$$

On the other hand, using both $\star$ -continuity and monotony, we have:

$$\begin{aligned} \left(\bigvee_{i \in I} x_i\right) \star \left(\bigvee_{i \in I} y_i\right) &= \bigvee_{j \in I} \left(\bigvee_{i \in I} x_i\right) \star y_j = \bigvee_{(i,j) \in I \times I} (x_i \star y_j) \leq \bigvee_{(i,j) \in I \times I} (x_{\max\{i,j\}} \star y_{\max\{i,j\}}) \\ &\leq \bigvee_{i \in I} (x_i \star y_i) \end{aligned}$$

The other direction is proved straightforwardly.

As it is seen in Lema 2, meet-continuity of the codomain-lattice implies composition-continuity. Moreover, we can prove the following proposition.

Proposition 3. If A is a set, L a complete lattice and $L^{A \times A}$ the set of L -valued relations on A , then the following is equivalent:

- (a) L is meet-continuous
- (b) $L^{A \times A}$ is left composition-continuous
- (c) $L^{A \times A}$ is right composition-continuous
- (d) $L^{A \times A}$ is composition-continuous

Proof. By Lemma 2, we have that ,meet-continuity of the codomen lattice implies the left and right continuity, as well as the continuity of $L^{A \times A}$, i.e. we have that (a) implies (b), (c) and (d).

We prove (b) $\Rightarrow$ (a):

Let $l \in L$ and let $\{m_i \mid i \in I\} \subseteq L$ be a chain in L .

Let P be the fuzzy relation for which $P(x, y) = l$ for all $(x, y) \in L \times L$. Let Q_i be the fuzzy relation for which $Q_i(x, y) = m_i$, for all $i \in I$ and all $(x, y) \in L \times L$. We have that

$$\begin{aligned} \left(\bigvee_{i \in I} Q_i\right)(x, y) &= \bigvee_{i \in I} m_i, \text{ for all } (x, y) \in A \times A \\ (P \circ Q_i)(x, y) &= l \wedge m_i, \text{ for all } i \in I \text{ and } (x, y) \in A \times A \\ \left(\bigvee_{i \in I} (P \circ Q_i)\right)(x, y) &= \bigvee_{i \in I} (l \wedge m_i) \\ (P \circ \left(\bigvee_{i \in I} Q_i\right))(x, y) &= l \wedge \bigvee_{i \in I} m_i, \text{ for all } (x, y) \in A \times A. \end{aligned}$$

Now, left composition-continuity implies

$$l \wedge \bigvee_{i \in I} m_i = \bigvee_{i \in I} (l \wedge m_i).$$

Analogously we prove that $(c) \Rightarrow (a)$, thus $(a) \Leftrightarrow (b) \Leftrightarrow (c)$.

What remains to be proved is that (d) implies (a) (or (b) or (c)). We prove $(d) \Rightarrow (b)$: suppose d is true, $P \in L^{A \times A}$ and $\{Q_i \mid i \in I\} \subseteq L^{A \times A}$ is a chain. Taking $P_i = P$ for all $i \in I$, and applying (d) we get:

$$P \circ (\bigvee_{i \in I} Q_i) = \bigvee_{i \in I} (P \circ Q_i).$$

4.2. General (in)equation

Let $\mathcal{X} = \{X_1, X_2, X_3, \dots\}$ be a countable set of variables.

We define the set of terms over $\mathcal{F}(A \times A)$ recursively:

- (i) All the variables from $\mathcal{X}$ are terms;
- (ii) All the relations from $\mathcal{F}(A \times A)$ are terms;
- (iii) If T_1 and T_2 are terms, then $T_1 \wedge T_2$, $T_1 \vee T_2$ and $T_1 \circ T_2$ are also terms.

We call the variables from $\mathcal{X}$ and the lattice-valued relations from $\mathcal{F}(A \times A)$ terms of zero complexity.

By $T(X_1, \dots, X_n)$ we will denote a term over $\mathcal{F}(A \times A)$, whose variables belong to $\{X_1, \dots, X_n\}$. Any such term define a function from $(L^{A \times A})^n$ to $L^{A \times A}$, mapping $(Y_1, \dots, Y_n)$ to the relation we get as the value of $T(Y_1, \dots, Y_n)$, i.e., the value of the term we get from T replacing any variable $X_i \in \{X_1, \dots, X_n\}$ occurring in $T(X_1, \dots, X_n)$ by the corresponding relation Y_i .

In the set $(L^{A \times A})^n$ of n -tuples of fuzzy relations, we define a componentwise order:

$$(R_1, \dots, R_n) \leq (R'_1, \dots, R'_n) \iff R_i \leq R'_i \text{ for all } i \in \{1, \dots, n\}$$

Clearly, for any family $\{(Y_1^i, \dots, Y_n^i) \mid i \in I\}$ in $(L^{A \times A})^n$, we have

$$\bigvee_{i \in I} (Y_1^i, \dots, Y_n^i) = (\bigvee_{i \in I} Y_1^i, \dots, \bigvee_{i \in I} Y_n^i)$$

Since the operations of infimum and supremum, as well as the composition of relations, are order-preserving relative to both arguments, by induction on the complexity of terms, we may prove the following lemma.

Lemma 4. *If $(R_1, \dots, R_n) \leq (R'_1, \dots, R'_n)$ in $(L^{A \times A})^n$ and $T(X_1, \dots, X_n)$ is a term, then $T(R_1, \dots, R_n) \leq T(R'_1, \dots, R'_n)$ in $L^{A \times A}$, i.e. the function*

$$T : (L^{A \times A})^n \rightarrow L^{A \times A},$$

defined by $T(X_1, \dots, X_n)$ is order-preserving.

Using this lemma, we prove the following proposition:

Proposition 4. *Let L be a complete, meet-continuous lattice and $\{(Y_1^i, \dots, Y_n^i) \mid i \in I\}$ a chain in $(L^{A \times A})^n$ and $T(X_1, \dots, X_n)$ a term over $L^{A \times A}$. Then $T(\bigvee_{i \in I} Y_1^i, \dots, \bigvee_{i \in I} Y_n^i) =$*

$$\bigvee_{i \in I} T(Y_1^i, \dots, Y_n^i).$$

Proof. We prove the assertion of the proposition by induction on the complexity of the term.

For the terms of the zero complexity, we have:

$$T(X_1, X_2, \dots, X_n) = X_i \in \mathcal{X} \quad \text{or} \quad T(X_1, X_2, \dots, X_n) = R \in L^{A \times A},$$

thus, the basis of the induction holds.

Let $T(X_1, \dots, X_n)$ be a more complex term than the zero complexity term, then $T(X_1, \dots, X_n) = T_1(X_1, \dots, X_n) * T_2(X_1, \dots, X_n)$, where $*$ may be $\wedge$ or $\vee$ or $\circ$. In all three cases, $L^{A \times A}$ is $*$ -continuous: by Proposition 1, it is meet-continuous, by Proposition 3, it is composition-continuous, while join-continuity is obvious. We assume that the assertion of the proposition holds for all the terms less complex than T . Since T_1 and T_2 are less complex, we have, by the induction hypothesis:

$$\begin{aligned} T_1\left(\bigvee_{i \in I} Y_1^i, \dots, \bigvee_{i \in I} Y_n^i\right) &= \bigvee_{i \in I} T_1(Y_1^i, \dots, Y_n^i) \\ T_2\left(\bigvee_{i \in I} Y_1^i, \dots, \bigvee_{i \in I} Y_n^i\right) &= \bigvee_{i \in I} T_2(Y_1^i, \dots, Y_n^i) \end{aligned}$$

By Lemma 4, $\{(T_1(Y_1^i, \dots, Y_n^i), T_2(Y_1^i, \dots, Y_n^i)) \mid i \in I\}$ is a chain.

$$\begin{aligned} T\left(\bigvee_{i \in I} Y_1^i, \dots, \bigvee_{i \in I} Y_n^i\right) &= T_1\left(\bigvee_{i \in I} Y_1^i, \dots, \bigvee_{i \in I} Y_n^i\right) * T_2\left(\bigvee_{i \in I} Y_1^i, \dots, \bigvee_{i \in I} Y_n^i\right) = \\ &= \bigvee_{i \in I} T_1(Y_1^i, \dots, Y_n^i) * \bigvee_{i \in I} T_2(Y_1^i, \dots, Y_n^i) \quad (\text{by Lemma 3}) \\ &= \bigvee_{i \in I} (T_1(Y_1^i, \dots, Y_n^i) * T_2(Y_1^i, \dots, Y_n^i)) = \bigvee_{i \in I} T(Y_1^i, \dots, Y_n^i) \end{aligned}$$

This general statement will enable us to prove the existence of a maximal solution in the general case of fuzzy relational equations and/or inequations, instead of just particular cases included in [19] and [20].

Theorem 1. *Let L be a complete meet-continuous lattice, $Z_1, \dots, Z_n \in L^{A \times A}$ and $T_1(X_1, \dots, X_n) = T_2(X_1, \dots, X_n)$ an equation over $L^{A \times A}$. If $(Y_1, \dots, Y_n)$ is a solution to that equation, such that $(Y_1, \dots, Y_n) \leq (Z_1, \dots, Z_n)$, then there exists a maximal solution - related to the componentwise order in $(L^{A \times A})^n$ - to the above equation contained in $(Z_1, \dots, Z_n)$. If the existing maximal solution is unique, it is also the greatest solution.*

Proof. Let $\{(Y_1^i, \dots, Y_n^i) \mid i \in I\}$ be a chain in $(L^{A \times A})^n$ which belongs to the set of the solutions to the equation $T_1(X_1, \dots, X_n) = T_2(X_1, \dots, X_n)$ contained in $(Z_1, \dots, Z_n)$. We have:

$$\begin{aligned} T_1(Y_1^i, \dots, Y_n^i) &= T_2(Y_1^i, \dots, Y_n^i) \text{ for all } i \in I, \text{ thus:} \\ \bigvee_{i \in I} T_1(Y_1^i, \dots, Y_n^i) &= \bigvee_{i \in I} T_2(Y_1^i, \dots, Y_n^i). \end{aligned}$$

By Proposition 4, we have

$$\begin{aligned} T_1\left(\bigvee_{i \in I} Y_1^i, \dots, \bigvee_{i \in I} Y_n^i\right) &= \bigvee_{i \in I} T_1(Y_1^i, \dots, Y_n^i) \text{ and} \\ T_2\left(\bigvee_{i \in I} Y_1^i, \dots, \bigvee_{i \in I} Y_n^i\right) &= \bigvee_{i \in I} T_2(Y_1^i, \dots, Y_n^i); \end{aligned}$$

$$\text{therefore } T_1\left(\bigvee_{i \in I} Y_1^i, \dots, \bigvee_{i \in I} Y_n^i\right) = T_2\left(\bigvee_{i \in I} Y_1^i, \dots, \bigvee_{i \in I} Y_n^i\right).$$

Thus, the chain has its union as an upper bound in the set of solutions to the equation $T_1 = T_2$ contained in $(Z_1, \dots, Z_n)$, since its union is also a solution to the equation contained

in $(Z_1, \dots, Z_n)$. Since this holds for any chain of solutions contained in $(Z_1, \dots, Z_n)$, applying Zorn's lemma, we get the assertion of the theorem.

Corollary 1. *If L is a complete meet-continuous lattice, every solvable relational equation over $L^{A \times A}$ of the type $T_1(X_1, \dots, X_n) = T_2(X_1, \dots, X_n)$ has a maximal solution.*

Analogously we prove that a maximal solution exists for any inequation as well. We use the term inequation to denote a constraint of the form $T_1 \leq T_2$ whose solutions are sought, in analogy with equations.

Theorem 2. *If L is complete and meet-continuous, $T_1(X_1, \dots, X_n) \leq T_2(X_1, \dots, X_n)$ is an inequation over $L^{A \times A}$ and $(Y_1, \dots, Y_n)$ its solution, such that $(Y_1, \dots, Y_n) \leq (Z_1, \dots, Z_n)$, there exists a maximal solution - relative to the componentwise order in $(L^{A \times A})^n$ - to the above inequation contained in $(Z_1, \dots, Z_n)$. If the existing maximal solution is unique, it is also the greatest solution.*

The following simple example demonstrates that in case L is not a meet-continuous lattice, we do not need to have a maximal solution to equations.

Example 1. Let L be a lattice in Figure 2. Let $A = \{d\}$ and let relations R and S be defined by: $R(d, d) = a$ and $S(d, d) = 0$. Now, we consider the equation $R \circ X = S$. We see that the relations B_i , defined by $B_i(d, d) = b_i$ ($i \geq 0$) are solutions to the equation. However, there is no maximal solution contained in the relation $R_1(d, d) = 1$, since $\bigvee_{i \geq 0} B_i = R_1$ and R_1 is not a solution of the equation.

In the same way, we prove that any system of L -valued relational equations and inequations has a maximal solution as well. Moreover, the following holds for any set of equations and inequations (possibly infinite):

Theorem 3. *If L is complete, meet-continuous, $\{T_1^i(X_1, \dots, X_n) = T_2^i(X_1^i, \dots, X_n^i) \mid i \in I\} \cup \{S_1^j(X_1, \dots, X_n) \leq S_2^j(X_1^j, \dots, X_n^j) \mid j \in J\}$ a set of equations and inequations over $L^{A \times A}$, and $(Y_1, \dots, Y_n)$ a solution to all of them, such that $(Y_1, \dots, Y_n) \leq (Z_1, \dots, Z_n)$, then there exists a maximal n -tuple of fuzzy relations - relative to the componentwise order in $(L^{A \times A})^n$, contained in $(Z_1, \dots, Z_n)$, such that it is a solution to all of the above equations and inequations. If the existing maximal relation is unique, it is the greatest such relation.*

The results we got generalize those from [19], where it was proved that there exists a maximal solution to each of the inequations: $P \circ Q \leq R$, $Q \circ P \leq R$, $P \circ Q \leq Q$, $Q \circ P \leq Q$, and $Q \circ Q \leq Q$, contained in a given fuzzy relation, where Q is the unknown. It was also proved that there exists a maximal solution to the inequation $P \circ Q \leq R$, where both P and Q are unknown. All these results are corollaries of Theorem 2 we have proved here. Moreover, when the fuzzy set equations are concerned, the composition-continuity we have proved here enables us to prove the existence of a maximal solution to the equations $P \circ Q = Q$, $Q \circ P = Q$, and $Q \circ Q = Q$, where Q is the unknown, or the existence of a maximal solution to the equation $P \circ Q = R$, where both P and Q are unknown. All these are corollaries of Theorem 1. In [19], the existence of a maximal solution was only proved for the corresponding inequations.

In [19], also the algorithms were developed for finding some closures of a fuzzy relation: the closure under the composition with a given fuzzy relation, and the transitive closure. These algorithms follow from a more general algorithm, introduced and proved in the following theorem.

Theorem 4. Let L be a complete, meet-continuous lattice, $T(X)$ a term and $R \in L^{A \times A}$. Let a sequence of fuzzy relations be defined recursively:

$$\begin{aligned} R_0 &= R \\ R_{n+1} &= T(R_n) \vee R_n. \end{aligned}$$

Then $\bigvee_{n \in \mathbb{N}} R_n$ is the least solution to the inequation $T(X) \leq X$ containing R .

Proof. The defined sequence is also a chain, since $R_n \subseteq T(R_n) \vee R_n = R_{n+1}$. Hence, $T(\bigvee_{n \in \mathbb{N}} R_n) = \bigvee_{n \in \mathbb{N}} T(R_n)$ follows from Proposition 4.

Therefore, $T(\bigvee_{n \in \mathbb{N}} R_n) = \bigvee_{n \in \mathbb{N}} T(R_n) \leq \bigvee_{n \in \mathbb{N}} R_{n+1} = \bigvee_{n \in \mathbb{N}} R_n$, thus $\bigvee_{n \in \mathbb{N}} R_n$ is a solution to the inequation $T(X) \leq X$.

On the other hand, any solution S to $T(X) \leq X$ containing R also contains R_n , for all $n \in \mathbb{N}_0$. We prove this by induction:

- (i) $R_0 = R \leq S$ by assumption
- (ii) $R_k \leq S$ implies $T(R_k) \leq T(S) \leq S$ (since T is order-preserving and S is a solution), thus $R_{k+1} = T(R_k) \vee R_k \leq S$, and the induction step holds.

Thus any solution to $T(X) \leq X$ containing R contains $\bigvee_{n \in \mathbb{N}} R_n$, thus $\bigvee_{n \in \mathbb{N}} R_n$ is the least solution to $T(X) \leq X$. $\square$

Taking $T(X) = X \circ P$ (or $T(X) = P \circ X$), we get an algorithm for finding the closure of a fuzzy relation under right (or left) composition with P , namely the one which was proved in [19].

Taking $T(X) = X \circ X$, we get an algorithm for finding the transitive closure of a given fuzzy relation, which was proved in [19].

Example 2. $L = (\mathbb{N} \vee \{\infty\})^{\mathbb{N} \vee \{\infty\}}$, where $\mathbb{N} \vee \{\infty\}$ is a lattice we get by adding ∞ to the poset $(\mathbb{N}, \leq)$, such that ∞ is the greatest element.

$$A = \mathbb{N}$$

$$P, Q, R, S \in L^{A \times A}$$

$$P(i, j)(n) = \begin{cases} n + j & i \geq j - 1 \\ n & i < j - 1 \end{cases}$$

$$S(i, j)(n) = n + i - j + 1$$

$$Q(i, j)(n) = \begin{cases} n + i & j = 1 \\ n + i + j & 1 < j \end{cases}$$

Here we take $\infty + k = \infty$, where $k \in \mathbb{R}$, so that in all the given functions $P(i, j)$, $Q(i, j)$ and $S(i, j)$, (for all $(i, j) \in \mathbb{N} \times \mathbb{N}$) ∞ are mapped to itself.

We construct the least solution to the inequation $P \circ X \wedge Q \leq X$ containing S , following the algorithm in Theorem 4:

$$R_0 = S$$

$$R_1(i, j)(n) = \begin{cases} n + i & j = 1 \\ \max\{n, n + i - j + 2\} & j > 1 \end{cases}$$

$$R_2(i, j)(n) = \begin{cases} n + i & j = 1 \\ n + i + 1 & j = 2 \\ \max\{n, n + i - j + 3\} & j > 2 \end{cases}$$

By induction on $k \in \mathbb{N}$ we prove:

$$R_k(i, j)(n) = \begin{cases} n + i & j = 1 \\ n + i + 1 & 1 < j \leq k \\ \max\{n, n + i - j + k + 1\} & j > k \end{cases}$$

Now we get the least solution:

$$R = \bigcup_{k \in \mathbb{N}} R_k$$

$$\text{We can check that } R(i, j)(n) = \begin{cases} n + i & j = 1 \\ n + i + 1 & j > 1 \end{cases}$$

We may further generalize Theorem 4, for a system of fuzzy inequations, to get an algorithm for the construction of the least solution to systems of relational inequations:

Theorem 5. Let $T_1(X_1, X_2, \dots, X_n), T_2(X_1, X_2, \dots, X_n), \dots, T_n(X_1, X_2, \dots, X_n)$ be terms and $(R_1, R_2, \dots, R_n)$ an n -tuple of fuzzy relations. If, by induction, we define the following sequence of n -tuples of fuzzy relations:

$$(Y_1^0, Y_2^0, \dots, Y_n^0) = (R_1, R_2, \dots, R_n)$$

$$(Y_1^{k+1}, \dots, Y_n^{k+1}) = (T_1(Y_1^k, \dots, Y_n^k) \vee Y_1^k, \dots, T_n(Y_1^k, \dots, Y_n^k) \vee Y_n^k),$$

then, $(\bigvee_{i \in \mathbb{N}_0} Y_1^i, \dots, \bigvee_{i \in \mathbb{N}_0} Y_n^i)$ is the least solution to the system

$$T_1(X_1, X_2, \dots, X_n) \leq X_1$$

$$T_2(X_1, X_2, \dots, X_n) \leq X_2$$

$$\dots$$

$$T_n(X_1, X_2, \dots, X_n) \leq X_n$$

containing $(R_1, R_2, \dots, R_n)$.

Proof. $(\bigvee_{i \in \mathbb{N}_0} Y_1^i, \dots, \bigvee_{i \in \mathbb{N}_0} Y_n^i)$ is a chain and it contains $(R_1, R_2, \dots, R_n)$ by definition.

We have that for $i \in \{1, \dots, n\}$:

$$T_i(Y_1^k, \dots, Y_n^k) \leq Y_i^{k+1}$$

$$\bigvee_{k \in \mathbb{N}_0} T_i(Y_1^k, \dots, Y_n^k) \leq \bigvee_{k \in \mathbb{N}_0} Y_i^{k+1} = \bigvee_{k \in \mathbb{N}_0} Y_i^k$$

$$T_i(\bigvee_{k \in \mathbb{N}_0} Y_1^k, \dots, \bigvee_{k \in \mathbb{N}_0} Y_n^k) = \bigvee_{k \in \mathbb{N}_0} T_i(Y_1^k, \dots, Y_n^k) \leq \bigvee_{k \in \mathbb{N}_0} Y_i^k$$

Thus, $(\bigvee_{k \in \mathbb{N}_0} Y_1^k, \dots, \bigvee_{k \in \mathbb{N}_0} Y_n^k)$ is a solution to the above system. To prove that it is the least solution to the system we take an arbitrary solution $(Z_1, \dots, Z_n)$ that is greater or

equal to $(R_1, \dots, R_n)$, and prove, by induction that (for $i \in \{1, 2, \dots, n\}$), Z_i contains Y_i^k , for all $k \in \mathbb{N}_0$.

$$Y_i^0 = R_i = Z_i$$

If $Y_i^k \leq Z_i$ for all $i \in \{1, 2, \dots, n\}$, then:

$Y_i^{k+1} = T_i(Y_1^k, \dots, Y_n^k) \vee Y_i^k \leq T_i(Z_1, \dots, Z_n) \vee Y_i^k \leq Z_i$ (since $(Z_1, \dots, Z_n)$ is a solution and $Y_i^k \leq Z_i$).

Now, Z_i is greater or equal to $\bigvee_{k \in \mathbb{N}_0} Y_i^k$, for $i \in \{1, 2, \dots, n\}$, and $(Z_1, \dots, Z_n)$ is greater or equal to $(\bigvee_{k \in \mathbb{N}_0} Y_1^k, \dots, \bigvee_{k \in \mathbb{N}_0} Y_n^k)$. $\square$

In the following theorem, we introduce an algorithm leading to the least solution of a wide class of systems of fuzzy relational equations.

Theorem 6. Let $X = \{X_1, \dots, X_n\}$ be a set of variables, and $T_1, T_2, \dots, T_n$ terms whose variables belong to X and the constants to the set of L -valued relations $L^{A \times A}$, where L is a complete, meet-continuous lattice. We define the sequence $(Y_1^k, \dots, Y_n^k)$ inductively:

$$Y_i^0 = 0, \text{ for all } i \in \{1, 2, \dots, n\};$$

$$Y_i^{k+1} = T_i(Y_1^k, Y_2^k, \dots, Y_n^k) \text{ for all } i \in \{1, 2, \dots, n\}.$$

Now $(\bigvee_{k \in \mathbb{N}} Y_1^k, \bigvee_{k \in \mathbb{N}} Y_2^k, \dots, \bigvee_{k \in \mathbb{N}} Y_n^k)$ is the least solution to the system:

$$X_1 = T_1(X_1, X_2, \dots, X_n)$$

$$X_2 = T_2(X_1, X_2, \dots, X_n)$$

....

$$X_n = T_n(X_1, X_2, \dots, X_n)$$

Proof. By induction, we prove that $(Y_1^k, \dots, Y_n^k) \leq (Y_1^{k+1}, \dots, Y_n^{k+1})$, for all $k \in \mathbb{N}$:

$$(Y_1^0, \dots, Y_n^0) = (0, 0, \dots, 0) \leq (Y_1^1, \dots, Y_n^1)$$

If $(Y_1^k, \dots, Y_n^k) \leq (Y_1^{k+1}, \dots, Y_n^{k+1})$, then for all $i \in \{1, \dots, n\}$:

$$Y_i^{k+1} = T_i(Y_1^k, \dots, Y_n^k) \leq T_i(Y_1^{k+1}, \dots, Y_n^{k+1}) = Y_i^{k+2},$$

Since $\{(Y_1^k, \dots, Y_n^k) \mid k \in \mathbb{N}_0\}$ is a chain, we may apply Proposition 4:

$$T_i(\bigvee_{k \in \mathbb{N}_0} Y_1^k, \bigvee_{k \in \mathbb{N}_0} Y_2^k, \dots, \bigvee_{k \in \mathbb{N}_0} Y_n^k) = \bigvee_{k \in \mathbb{N}_0} T_i(Y_1^k, \dots, Y_n^k) = \bigvee_{k \in \mathbb{N}_0} Y_i^{k+1} = \bigvee_{k \in \mathbb{N}_0} Y_i^k$$

Thus, $(\bigvee_{k \in \mathbb{N}} Y_1^k, \bigvee_{k \in \mathbb{N}} Y_2^k, \dots, \bigvee_{k \in \mathbb{N}} Y_n^k)$ is a solution.

On the other hand, if $(S_1, \dots, S_n)$ is a solution, then we prove by induction that $(Y_1^k, \dots, Y_n^k) \leq (S_1, \dots, S_n)$:

$$(Y_1^0, \dots, Y_n^0) = (0, 0, \dots, 0) \leq (S_1, \dots, S_n).$$

If $(Y_1^k, \dots, Y_n^k) \leq (S_1, \dots, S_n)$, then by monotony of T_i we have:

$$Y_i^{k+1} = T_i(Y_1^k, \dots, Y_n^k) \leq T_i(S_1, \dots, S_n) = S_i, \text{ thus } (Y_1^{k+1}, \dots, Y_n^{k+1}) \leq (S_1, \dots, S_n)$$

Thus, $(Y_1^k, \dots, Y_n^k) \leq (S_1, \dots, S_n)$ for all $k \in \mathbb{N}_0$ and, therefore:

$$(\bigvee_{k \in \mathbb{N}_0} Y_1^k, \dots, \bigvee_{k \in \mathbb{N}_0} Y_n^k) \leq (S_1, \dots, S_n).$$

Since $(\bigvee_{k \in \mathbb{N}_0} Y_1^k, \dots, \bigvee_{k \in \mathbb{N}_0} Y_n^k)$ is less than or equal to any solution to the system, it is the least solution.

Example 3. Let L be the lattice in Figure 3. It is meet-continuous, since it is algebraic. Namely, all the elements of the form a_k , where $k \in \mathbb{N}$ are compact: if $\{c_j \mid j \in J\}$ is a family of elements from L and $a_k \leq \bigvee_{j \in J} c_j$, there must be some element of the form a_k among c_j (else $\bigvee_{j \in J} c_j \leq b$); also, there must be some a_i or b_i with $i \geq k$ (or else $\bigvee_{j \in J} c_j \leq a_{k-1}$), or b , or 1 ; in the first case $a_k \leq a_i$; in the second case, $a_k \leq b_i \vee a_k$; in the third case $a_k \leq b \vee a_k$; in the fourth case $a_k \leq 1$. Similarly, we prove that a is compact. It is easy to prove that elements of the form b_k are compact. As for b and 1 , they are supremum of the compact elements. Thus L is algebraic and also meet-continuous.

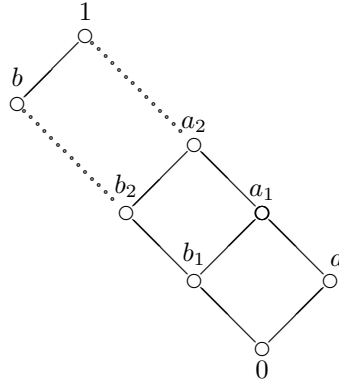


Fig. 3. The codomain lattice in Example 3

Let $A = N$. We define the following fuzzy relational equations on A , with L as the codomain lattice:

$$\begin{aligned} P(i, j) &= b_{i+j} \\ Q(i, j) &= \begin{cases} b & \text{if } j - i \leq 1, \\ 0 & \text{otherwise.} \end{cases} \\ R(i, j) &= \begin{cases} 1 & \text{if } i - j \leq 1, \\ 0 & \text{otherwise.} \end{cases} \\ S(i, j) &= a_{i+j} \end{aligned}$$

Now we consider the following system of equations:

$$\begin{aligned} X &= P \vee (Q \circ Y) \\ Y &= S \vee (X \circ R) \end{aligned}$$

We construct the least solution to the system, using the above algorithm:

$$\begin{aligned} (X_0, Y_0) &= (0, 0) \\ (X_1, Y_1) &= (P, S) \\ X_2(i, j) &= b_{i+j+1}; Y_2(i, j) = a_{i+j+1}. \end{aligned}$$

By induction on n , we prove that for $n \geq 2$ we have:

$$X_n(i, j) = b_{i+j+n-1}; Y_n(i, j) = a_{i+j+n-1}.$$

The basis for $n = 2$ we already have. If $X_k(i, j) = b_{i+j+k-1}; Y_k(i, j) = a_{i+j+k-1}$, since $X_{k+1} = P \vee (Q \circ Y_k)$ and $Y_{k+1} = S \vee (X_k \circ R)$, it is easy to see that:

$$X_{k+1}(i, j) = b_{i+j+k}; Y_{k+1}(i, j) = a_{i+j+k}.$$

Now, the least solution to the system is $(X', Y') = \bigvee_{k \in \mathbb{N}} (X_k, Y_k)$. Obviously, $X'(i, j) = b$ and $Y'(i, j) = 1$ for all $(i, j) \in \mathbb{N} \times \mathbb{N}$. Due to Theorem 5, we don't have to check that (X', Y') is a solution, or to prove that it is the least one.

5. Potential Real-World Applications

Although the present study is primarily theoretical, the obtained results have several potential applications. Fuzzy relational equations and inequations provide a flexible mathematical framework for modeling systems in which relationships between objects are imprecise, vague, or partially unknown. In such settings, relations are described by degrees of association taking values in a complete lattice or another ordered algebraic structure. This makes them suitable for modeling real-world problems characterized by uncertainty, incompleteness, or incomparable information.

The study of both equations and inequations is essential, as they capture different types of constraints: equations typically express consistency or equilibrium conditions, while inequations naturally arise when information is partial or when constraints are admissibility conditions.

One important area of application is decision support and multi-criteria decision making [2,12], where preferences and expert judgments are often expressed by fuzzy values. Consistency requirements between alternatives and criteria can be modeled using fuzzy relational equations, while inequations describe admissible or acceptable decisions. Similar formulations appear in recommendation systems and group decision processes.

Another application domain is knowledge representation and data analysis, particularly in connection with formal concept analysis [1,7]. Here, fuzzy relations describe graded associations between objects and attributes. Fuzzy relational equations arise when inferring missing data or determining closures, whereas inequations model approximate implications and soft constraints, which are especially relevant in the presence of noisy data.

Further applications can be found in control theory and systems modeling [5,15], where fuzzy relational equations are used to represent input–output relations and rule bases, while inequations describe admissible operating regions and robustness requirements. Additional applications include image processing, pattern recognition, and information retrieval, where similarity and relevance relations are inherently fuzzy. To illustrate the applicability of the framework, we consider the following simple example, inspired by models in fuzzy decision making (see, e.g., [2,12]).

Let $A = \{a_1, a_2\}$ be a set of alternatives (e.g., two treatment options or products). Suppose that this set is evaluated with respect to a set of criteria $B = \{b_1, b_2\}$ (e.g., effectiveness and cost). Let L be a finite lattice and let $R : A \times B \rightarrow L$ describe the degree to which each alternative satisfies each criterion, and $X : B \rightarrow L$ represents the importance (weights) of the criteria.

The overall evaluation $Y : A \rightarrow L$ of the alternatives can be modeled by the fuzzy relational equation $R \circ X = Y$ where $\circ$ denotes the standard (sup–inf) composition.

In practice, the desired evaluation Y may only be partially specified or constrained (e.g., one alternative should be at least as good as a given threshold). This leads naturally to a fuzzy relational inequation of the form $R \circ X \geq Y$ which describes all admissible weight vectors X consistent with the requirements. The problem then becomes to determine all solutions X , or in particular the maximal and least solutions. The theoretical results developed in this paper guarantee the existence of maximal solutions under appropriate conditions on the lattice L , and provide constructive procedures for obtaining the least solution.

This example illustrates how practical problems can be reduced to fuzzy relational equations and inequations. The main results of this paper ensure the existence of maximal solutions and provide terminating procedures for computing the least solution, thereby offering both theoretical guarantees and computational tools.

6. Conclusion

We generalized the notion of meet-continuity of a complete lattice and proved some lattice-theoretic properties. We proved the equivalence between meet-continuity of the codomain lattice and different kinds of composition-continuity in the lattice of L -valued relations.

The existence of a maximal solution to general L -valued relational equations and inequations and their systems was proved in the case L is a complete, meet-continuous lattice. Moreover, there exists a maximal L -valued relational equation that is a solution to all the equations from a given set, which may be infinite as well.

An algorithm is developed for the construction of the least solution of a wide class of equations and inequations, in the case of a complete, meet-continuous lattice.

In the future, we shall try to find algorithms for the construction of a maximal solution or the whole set of maximal solutions, in the case of a complete, meet-continuous codomain lattice.

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References

1. Alcalde, C., Burusco, A., Díaz-Moreno, J.C., Medina, J.: Fuzzy concept lattices and fuzzy relation equations in the retrieval processing of images and signals. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems* 25(Suppl. 1), 99–120 (2017)
2. Aliev, R., Guirimov, B., Huseynov, O., Aliyev, R.: Z-relation equation-based decision making. *Expert Systems with Applications* 184, 115387 (2021)

3. Cornejo, M., Lobo, D., Medina, J., De Baets, B.: Bipolar equations on complete distributive symmetric residuated lattices: The case of a join-irreducible right-hand side. *Fuzzy Sets and Systems* 442, 92–108 (2022)
4. Davey, B.A., Priestley, H.A.: *Introduction to Lattices and Order*. Cambridge University Press, 2 edn. (2002)
5. De Baets, B.: Analytical solution methods for fuzzy relational equations. In: *Fundamentals of Fuzzy Sets*. The Handbooks of Fuzzy Sets Series, Springer, Boston (2000)
6. Díaz-Moreno, J., Medina, J., Turunen, E.: Minimal solutions of general fuzzy relation equations on linear carriers, an algebraic characterization. *Fuzzy Sets and Systems* 311, 112–123 (2017)
7. Ganter, B., Wille, R.: *Formal Concept Analysis: Mathematical Foundations*. Springer (2024)
8. Gierz, G., Hofmann, K.H., Keimel, K., Lawson, J.D., Mislove, M., Scott, D.S.: *Continuous Lattices and Domains*, vol. 93. Cambridge University Press (2003)
9. Gierz, G., Hofmann, K.H., Keimel, K., Lawson, J.D., Mislove, M., Scott, D.S.: *A Compendium of Continuous Lattices*. Springer (2012)
10. Goguen, J.: L-fuzzy sets. *Journal of Mathematical Analysis and Applications* 18, 145–174 (1967)
11. Hohle, U.: Upper semicontinuous I-fuzzy sets and applications. *Fuzzy Sets and Systems* 27, 217–237 (1987)
12. Kahraman, C. (ed.): *Fuzzy Multi-Criteria Decision Making: Theory and Applications with Recent Developments*. Springer (2008)
13. Keimel, K., Lawson, J.: Continuous and completely distributive lattices. In: *Continuous and Completely Distributive Lattices*, pp. 5–53. Springer, Cham (2014)
14. Pavelka, J.: On fuzzy logic ii. enriched residuated lattices and semantics of propositional calculi. *Mathematical Logic Quarterly* 25, 119–134 (1979)
15. Peeva, K., Kyosev, Y.: *Fuzzy Relational Calculus*. World Scientific (2005)
16. Rachunek, J., Slezak, V.: Bounded dually residuated lattice ordered monoids as a generalization of fuzzy structures. *Mathematica Slovaca* 56(2), 223–233 (2006)
17. Rodabaugh, S.: Powerset operator foundations for poslat-fuzzy set theories. *Fuzzy Sets and Systems* 63, 129–144 (1994)
18. Sanchez, E.: Resolution of composite fuzzy relation equations. *Information and Control* 30, 38–48 (1976)
19. Stepanovic, V.: Fuzzy set inequations and equations with a meet-continuous codomain lattice. *Journal of Intelligent and Fuzzy Systems* 34(6), 4009–4021 (2018)
20. Stepanovic, V.: Two sufficient conditions for having a maximal solution to fuzzy relational equations. In: *Computational Intelligence and Mathematics for Tackling Complex Problems* 6, pp. 19–26. Springer, Cham (2025)
21. Stepanovic, V., Tepavčević, A.: Fuzzy sets (in)equations with a complete codomain lattice. *Kybernetika* 58, 145–162 (2022)
22. Stepanovic, V., Tepavčević, A.: A note on the solutions to lattice-valued relational equations and inequations. *Acta Polytechnica Hungarica* 22(12), 27–46 (2025)
23. Tarski, A.: On the calculus of relations. *The Journal of Symbolic Logic* 6(3), 73–89 (1941)
24. Tarski, A.: A lattice-theoretical fix point theorem and its applications. *Pacific Journal of Mathematics* 5, 285–309 (1955)
25. Šešelja, B., Tepavčević, A.: *Weak Congruences in Universal Algebra*. Mathematical Institute of the Serbian Academy of Sciences and Arts, Novi Sad (2001)

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